Meghan's Challenge Problems, Weeks 1-5

$\boxed{\mathbf{1}}$ Prove that, in a non-degenerate right triangle (i.e. all side lengths are greater than 0), the length of the hypotenuse is less than the sum of the lengths of the two remaining sides.

$\boxed{\mathbf{2}}$ Prove that the set $\{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable. (If a set is *countable* then you can make a (possibly infinite) numbered list of all the elements. You want to show that you cannot make such a list containing *all* of the real numbers between 0 and 1.)

$\boxed{\mathbf{3}}$ In the Tower of Hanoi game, a player is given three rods (that are standing upright) and n disks which all have different sizes and can be slid onto the rods. The player begins with the disks stacked on the leftmost rod in ascending order by size (The largest disk is on the bottom, the second largest disk is directly on top of it, the third largest disk is directly on top of that, and so on.) The player's goal is to move all of the disks to the third rod. However, there are two rules. Firstly, the player is only allowed to move one disk at a time. Secondly, the player is never allowed to put any disk on top of a smaller disk. Prove that you can always solve the Tower of Hanoi game in $2^n - 1$ moves, where n is the number of disks.

$\boxed{\mathbf{4}}$ Let $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$ with $F(0) = F(1) = 1$. Use induction to prove that for all $n \geq 0$, $\sum_{i=0}^{n} F(2i) = F(2n+1)$. (Hint: At some point, you may need to use the fact that you can re-write the recursive formula for $F(n)$ as either $F(n-1) = F(n) - F(n-2)$ or $F(n-2) = F(n) - F(n-1)$.)

$\boxed{\mathbf{5}}$ Let $D(m,n)$ be the function defined by $D(m,0) = D(0,n) = 1$ and, for $m,n \geq 1$, $D(m,n) = D(m,n-1) + D(m-1,n-1) + D(m-1,n)$. These numbers are called the Delannoy numbers, and count the number of different ways to make a path from $(0,0)$ to (m,n) using only horizontal steps that go right one unit, vertical steps that go up one unit, and diagonal steps that go up and right one unit. Can you figure out why this recursive formula indeed counts the number of such paths? Prove (using the recursive formula) that $D(m,n) \geq \sum\limits_{k=0}^{\min\{m,n\}} 2^k$ for all $m,n \geq 0$.

6 A group of people with assorted eye colors live on an island. They are all perfect logicians (i.e. if a conclusion can be logically deduced, then they will do so instantly). In addition, they all know that everyone else is a perfect logician. They are not, however, allowed to communicate.

On this island, there are 100 blue-eyed people, 100 brown-eyed people, and a Guru (who happens to have green eyes). Each person can see everyone else's eye color, but does not know their own eye color. Every night at midnight, a ferry stops at the island. Any islanders who have figured out the color of their own eyes can then leave the island. One day at noon, the guru is allowed to say one sentence. She says "I can see someone who has blue eyes." Who leaves the island, and on what night? (source: `https://xkcd.com/blue_eyes.html`)