CS 173 Problems

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September 26, 2018

1 Contradiction

1. A finite set S of points in the plane has the property that any line through two of them passes through a third. Prove that all points in S lie on a line.

2. Five points are placed inside a equilateral triangle of side 1. Prove that two of these points must be at a maximum distance of $1/2$ from each other.

3. Prove that out of a party of 6 people, there exists a group of 3 mutual friends or a group of 3 mutual non-friends.

4. We say that a point $P = (x, y)$ in the Cartesian plane is rational if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathbb{Q}^2$. An equation $F(x, y) = 0$ is said to have a rational point if there exists $(x_0, y_0) \in \mathbb{Q}$ such that $F(x_0, y_0) = 0$. For example, the curve $x^2+y^2-1 = 0$ has rational point $(x_0, y_0) = (1, 0)$. Show that the curve $x^2 + y^2 - 3 = 0$ has no rational points.

5. Find all solutions in non-negative integers of $x^3 + 2y^3 = 4z^3$.

6. Prove that there is no quadruple of positive integers (x, y, z, u) satisfying $x^2 + y^2 = 3(z^2 + u^2)$.

2 Induction

1. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.

2. Suppose every road in a country is one-way. Every pair of cities in the country is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

3. For all $n \in \mathbb{Z}$, prove that $f(n) = g(n)$, where:
   $f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n+1} - \frac{1}{2n}$, and $g(n) = \frac{1}{n+1} + \cdots + \frac{1}{2n}$.

4. Consider n distinct lines in the plane such that no two are parallel and no three lines intersect at a common point. Into how many different regions is the plane divided?

5. Prove that it is always possible to cover a checkerboard of size $2^n n \times 2^n$ with one squared removed using L-shaped triominos (3 squares in the shape of an "L").

6. Prove that $(n+1)(n+2)\cdots 2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$ for all $n \in \mathbb{Z}$.

7. Prove that a convex n-gon has $n(n-3)/2$ diagonals.