### Problem 1

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**Total out of 30 points**
Problem 1: 10 points. Let the function \( F(n) : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) be defined by

- \( F(1) = 5 \)
- \( F(n) = 2F(n - 1) \) for \( n \geq 2 \)

Prove, using induction, that \( F(n) = 5 \times 2^{n-1} \) all \( n \geq 1 \).

Solution:

We verify that the statement holds true for \( n = 1 \). Thus, \( F(1) = 5 \) and \( 5 \times 2^{1-1} = 5 \times 2^0 = 5 \times 1 \). Hence, the base case holds.

Our Inductive Hypothesis (I.H.) is that \( \exists K \geq 1 \) s.t. \( \forall n \in \{1, 2, \ldots, K\}, F(n) = 5 \times 2^{n-1} \). We now show that \( F(K + 1) = 5 \times 2^K \). Since \( K + 1 \geq 2 \), \( F(K + 1) = 2F(K) \), by definition. Then we apply the inductive hypothesis, and get

\[
F(K + 1) = 2F(K) = 2 \times 5 \times 2^{K-1} = 5 \times 2^K.
\]

This is what we wanted to prove.

Since \( K \) was arbitrary, the theorem is proven.
Problem 2: 10 points. Simplify the following expression so that it has no →s and no parentheses. Explain all your steps.

\((\neg A \rightarrow \neg B) \rightarrow A\)

Solution.

There are many ways to do this problem, including using truth tables. We will do it by simplifying each part.

We begin by simplifying \((\neg A \rightarrow \neg B)\) (the left hand side of the expression).

We know that \(P \rightarrow Q \equiv \neg P \lor Q\), so we obtain

\[(\neg A \rightarrow \neg B) \equiv (A \lor \neg B)\]

Thus,

\[[\neg A \rightarrow \neg B) \rightarrow A] \equiv [(A \lor \neg B) \rightarrow A]\]

We apply the rule for how to remove the → again, and obtain

\[[A \lor \neg B) \rightarrow A] \equiv [\neg(A \lor \neg B) \lor A]\]

We now have to use De Morgan’s laws to remove the parentheses. Note that \(\neg(A \lor \neg B) \equiv (\neg A \land \neg \neg B) \equiv (\neg A \land B)\). Hence, we continue the reduction and have

\[[A \lor \neg B) \rightarrow A] \equiv [\neg(A \lor \neg B) \lor A] \equiv [(\neg A \land B) \lor A]\]

which we write more simply as

\[[A \lor \neg B) \rightarrow A] \equiv [(\neg A \land B) \lor A]\]

We now have to use the distribution property to simplify the right hand side. We obtain

\[[\neg A \land B) \lor A] \equiv [(\neg A \lor A) \land (B \lor A)]
Since \( \neg A \lor A \equiv T \), the above expression simplifies to \( B \lor A \), which is equivalent to \( A \lor B \). Hence, we have shown

\[
[(A \lor \neg B) \rightarrow A] \equiv (A \lor B)
\]

Since the parentheses around \( A \lor B \) aren’t necessary (and are only there to help you parse the sentence), what we have shown is

\[
[(\neg A \rightarrow \neg B) \rightarrow A] \equiv A \lor B
\]

We now check that our answer is correct, because it’s easy to make mistakes. If \( A \) is true, then the right hand side of the expression \((\neg A \rightarrow \neg B) \rightarrow A\) is true, which sets the whole expression to true. If \( A \) is false, then the right hand side is false, and so the expression is true if and only if the left hand side is false. But since \( A \) is false, to make the expression \( \neg A \rightarrow \neg B \) false, we must have \( B \) being true. Hence, we have verified that the expression is true if and only if \( A \) is true or \( B \) is true, which is what we had derived through simplifying the expressions.
Problem 3: 10 points. Consider the following 2CNF formula:

\[(\neg A \lor \neg B) \land (B \lor C)\]

- Determine whether this formula is a tautology, satisfiable but not a tautology, or not satisfiable (and prove your answer correct). (5pts)

Solution: the expression is satisfiable but not a tautology. To satisfy it, you can set \(A = F\) and \(B = C = T\). But it is not a tautology, because setting \(A = B = C = F\) makes the expression false.

- Show the directed graph associated to this 2CNF formula (as described in the class presentation). (5 pts)

Solution: There are three variables so six vertices, and two clauses so four directed edges. The vertices are named after the variables and their negations. The directed edges are:

\[A \rightarrow \neg B\]
\[B \rightarrow \neg A\]
\[\neg B \rightarrow C\]
\[\neg C \rightarrow B\]

If you draw the graph, you will find two vertex-disjoint directed paths: \(A \rightarrow \neg B \rightarrow C\) and \(\neg C \rightarrow B \rightarrow \neg A\).