CS 173 (B) Fall 2015
Review Questions for Final Exam

Comments: The final exam is likely to be based on these questions (or using the same skills). Therefore, if you can do these questions well, you will be able to do well on the final.

I will only include at most one question labelled with a (*) on the final exam (and it will count at most 5% of the exam). Hence, focus on the other questions.

You should also look over all the homeworks, reading quizzes, and examlets; any question in any of these is just as likely to show up on the final exam as the ones in this set.

1 Logic Expressions

Basic knowledge:

1. Know what it means to say that a logical expression is a tautology, or is satisfiable, or is not satisfiable.
2. Know what 2CNF refers to, and know the graph algorithm to determine if a 2CNF formula is satisfiable.
3. Be able to simplify logical expressions, such as

$$(\neg A \to \neg B) \to A$$

   so that there are no implications and no parentheses.

2 Big-oh

Basic knowledge:

1. Give the definition of big-oh (i.e., what it means to say that f is $O(g)$).
2. Be able to prove or disprove that f is $O(g)$ (given two functions f and g).
3. Be able to quickly compare functions and determine if one is big-oh of the other.

Problems:

1. Let $f(n) = n^{3/2}$ and $g(n) = \frac{n^2}{\log n}$. Is either function big-oh of the other? Prove your claim.
2. Suppose $f : \mathbb{N} \to Z^+$ is defined by $f(n) = 0$ if n is odd, and $f(n) = n$ if n is even. Let $g(n) = 1$ for all n, and let $h(n) = n$. Determine the veracity of each of the following statements:
   - f is $O(g)$
   - f is $O(h)$
   - g is $O(f)$
   - g is $O(h)$
   - h is $O(f)$
   - h is $O(g)$
3. Find constants C_1 and C_2 so that $f(n) \leq C_1 g(n)$ for all $n \geq C_2$ where $f(n) = 500n$ and $g(n) = n^2$.
4. Let $f(n) = n!$, $g(n) = 2^n$, and $h(n) = n^{n/2}$. For each pair of functions, determine which is big-oh of the other.

3 Sets, Relations, Functions, and Counting

Basic knowledge:

1. Be able to do the elementary counting tasks (e.g., count the number of ways you can put n people in a row, or the number of ways you can pair up n boys and n girls, or pick n items out of m if the order does not matter, or pick n items out of n if the order does matter, etc.).
2. Review the Inclusion-Exclusion principle and be able to use it.
3. Know the basic algorithmic techniques for counting: (1) algorithmically generate the set in a decision tree (and count the leaves of the tree), (2) count the complement, (3) perform a case analysis, (4) write the set as a union of sets, and apply the Inclusion-Exclusion principle. Be aware that some counting problems require more than one of these techniques.
4. Know all the definitions of the properties about relations (e.g., reflexive, irreflexive, symmetric, anti-symmetric, transitive). Know what a partial order has to satisfy. Know what an equivalence relation has to satisfy.
5. Know the definitions of injection (1-1 function), surjection (onto function), and bijection.
6. Know what the domain and co-domain refer to.
7. Know the definition of infinite, finite, countable, and uncountable sets.
8. Know set-builder notation.

Problems:

1. Let $A = \{f : \{0,1\} \to \{0,1,2\}\}$. Answer the following questions:
   - What is $|A|$?
   - Does A contain any 1-1 functions? If so, how many? (List them)
   - Does A contain any surjective (onto) functions? If so, how many? (List them)
   - Does A contain any bijections? If so, how many? (List them)
2. Let $A = \{f : \{0,1,2\} \to \{0,1\}\}$. Answer the following questions:
   - What is $|A|$?
   - Does A contain any 1-1 functions? If so, how many? (List them)
   - Does A contain any surjective (onto) functions? If so, how many? (List them)
   - Does A contain any bijections? If so, how many? (List them)
3. Let $A = \{f : \{0,1,2\} \to \{0,1,2,3\}\}$. Answer the following questions:

- What is $|A|$?
- Does A contain any 1-1 functions? If so, how many?
- Does A contain any surjective (onto) functions? If so, how many?
- Does A contain any bijections? If so, how many?

4. Let $A = \{f : \{0, 1, 2, 3\} \to \{0, 1, 2\}\}$. Answer the following questions:

   - What is $|A|$?
   - Does A contain any 1-1 functions? If so, how many?
   - Does A contain any surjective (onto) functions? If so, how many?
   - Does A contain any bijections? If so, how many?

5. Let X be the set of finite non-empty subsets of Z^+ and let $Y = \mathbb{P}(Z^+)$.

   - Draw the Venn Diagram of the sets X and Y.
   - Give an example of an element of $Y \setminus X$.
   - Prove that X is infinite.
   - Prove that Y is infinite.
   - Prove that X is countable.
   - Prove that Y is uncountable.

6. Let $X_n = \{A \subseteq Z^+ : |A| \leq n\}$ (i.e., X_n denotes the set of subsets of Z^+ that have at most n elements).

   - Give an element of X_1. Is X_1 finite?
   - Give an element of X_2. Is X_2 finite?
   - Let $X = \bigcup_n X_n$ Is $X = \mathbb{P}(Z^+)$?
   - Now let $Y_n = \mathbb{P}(\{1, 2, \ldots, n\})$, and let $Y = \bigcup_n Y_n$. Is there any element of Y that is infinite? Is $Y = \mathbb{P}(Z^+)$? Does $\mathbb{P}(Z^+)$ contain any infinite sets?
   - Draw the Venn Diagram for X and Y, and show whether the areas are empty or non-empty. If you think an area is non-empty, give an element in the area; if you think an area is empty, explain why.

7. Consider the relation R defined by $R = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z} : x^2 = y^2\}$. Is R symmetric? anti-symmetric? transitive? reflexive? irreflexive? Is R a partial order? Is R an equivalence relation?

8. Consider the relation R defined by $R = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z} : x^2 \leq y^2\}$. Is R symmetric? anti-symmetric? transitive? reflexive? irreflexive? Is R a partial order? Is R an equivalence relation?

9. Consider the relation R defined by $R = \{(x, y) : x \in \mathbb{N}, y \in \mathbb{N} : x^2 = y^2\}$. Is R symmetric? anti-symmetric? transitive? reflexive? irreflexive? Is R a partial order? Is R an equivalence relation?

10. Let $f : \mathbb{P}(\mathbb{N}) \to \mathbb{P}(\mathbb{N})$ be defined by $f(A) = A \cup \{0\}$.

- What is $f(\emptyset)$?
- What is $f(\{0, 1\})$?
- What is $f(\{1, 2\})$?
- Consider $X = \{A \subseteq \mathbb{N} : f(A) = A\}$. Give an element of X. Give an element of $\mathbb{P}(\mathbb{N}) \setminus X$.

4 Graphs

Basic knowledge:

1. Know the definition of vertex cover, dominating set, clique, independent set, matching, chromatic number, Eulerian walk, Eulerian circuit, Hamiltonian path, and Hamiltonian cycle. Know what a bipartite graph is.
2. Know how to describe a graph, and how to read the set-theoretic definition of a graph.
3. Know what a simple graph is.
4. Know adjacency matrix and adjacency list notation, and how to use it with directed and undirected graphs.
5. Know Kruskal's algorithm for Minimum Spanning Tree.
6. Know the 2-approximation for vertex cover, and why it works.
7. Know BFS (Breadth-First-Search), and how to compute a BFS tree.
8. Know the meaning of each of the following terms: $K_n, K_{m,n}, P_n, C_n, W_n$.
9. Know how to describe a real world problem using graph-theoretic terminology.

Problems:

1. For each of the following graphs, find a minimum vertex cover, minimum dominating set, maximum clique, maximum independent set, maximum matching, and chromatic number. Also determine if the graph has an Eulerian walk, an Eulerian circuit, a Hamiltonian path, or a Hamiltonian cycle.
   - K_5
   - K_6
   - $K_{2,3}$
   - $K_{3,3}$
   - C_5
   - C_6
   - W_5
   - W_6
   - P_5
   - P_6

- Draw a few graphs with at most 5 or 6 vertices, and see if you can answer the questions given above.

2. Let A_n be the set of simple graphs on n distinctly labelled vertices $v_1, v_2, \ldots, v_n$. We want to count the number of elements in A_n, so you should know that we will consider the graph with three vertices and two edges $(v_1, v_2), (v_2, v_3)$ to be different from the graph with the same three vertices but edges (v_1, v_2), (v_1, v_3). Answer the following questions:
   - Draw all graphs in A_3
   - What is $|A_n|$?
3. Let B_{2n} be the set of bipartite graphs with $v_1, v_2, \ldots, v_n$ on one side and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ on the other side. What is $|B_{2n}|$? Does every graph in B_{2n} have a Hamiltonian path? Does every graph in B_{2n} have a Hamiltonian cycle? Answer these questions for the complete bipartite graphs in B_{2n}.
4. How many Hamiltonian paths exist in the graph K_n? (List them for $n = 3$ and $n = 4$.)
5. How many Hamiltonian cycles exist in the graph K_n? (List them for $n = 3$ and $n = 4$.)
6. Suppose G is a graph with 200 vertices, and I find a maximal matching M in G that has 20 edges. What can you tell me about:
   - The size of the smallest vertex cover in G?
   - The size of the maximum matching in G?
   - The size of the largest independent set in G?
7. Give an example of a graph where the number of vertices in the smallest dominating set is different from the number of vertices in the smallest vertex cover.
8. Give an example of a graph where the number of vertices in the smallest dominating set is equal to the number of vertices in the smallest vertex cover.
9. Let $G = (V, E)$ be a graph, and let $V_0 \subseteq V$. Prove that V_0 is an independent set of G if and only if $V \setminus V_0$ is a vertex cover of G.
10. Suppose that G is a graph and V_0 is a subset of the vertices that is simultaneously a vertex cover and an independent set. Prove that G is bipartite.
11. Prove that every graph on n vertices that has at least n edges must have a cycle.
12. Suppose you are at Thanksgiving, and your parents have told you and your brother to take care of all the kids in the extended family. Yep, all your cousins and second cousins are there, and not all of them get along. You want to separate the kids into two rooms (one room for you, and one room for your brother), so that the kids in each room get along. You don't know if this is possible.
    - Describe this as a graph-theoretic problem. What are the vertices? What are the edges? What are you trying to construct?

- Suppose your family has 10 girls and n boys, and the girls get along with everyone but the boys all fight with each other. What is the largest value for n where you can solve your problem?

13. Same Thanksgiving problem, but now you'll settle for partitioning the kids into two rooms so that the total number of pairs of kids who don't get along and are in the same room is minimized.

    - Describe this as a graph-theoretic problem. What are the vertices? What are the edges? What are you trying to construct? (This is an optimization problem, so describe this carefully.)
    - Suppose your family has 10 girls and n boys, and the girls get along with everyone but the boys all fight with each other. If $n = 3$, what does your graph look like, and what does your solution look like?
    - Suppose your family has 2 girls and 2 boys, the girls do not get along with each other, the boys do not get along with each other, but all girls get along with all boys. What does your graph look like? What does your solution look like?
    - Suppose your family has 3 girls and 3 boys, and the girls do not get along with each other, the boys do not get along with each other, but all the girls get along with all the boys. What does your graph look like? What does your solution look like?

5 Induction Proofs

Basic skills.

1. You should know both weak and strong induction. If you have trouble with strong induction, use weak induction – but be careful, because it isn't always valid.
2. You should be able to write a clear inductive hypothesis.
3. You should use your variables properly.
4. If you have trouble understanding induction and why it works, do your proof using contradiction. Then go back and think about how to prove your statement using induction.
5. Don't forget the base case. And remember that there can be more than one base case!

Problems:

1. Let the function $F(n) : Z^+ \rightarrow Z^+$ be defined by
   - $F(1) = 5$
   - $F(2) = 6$
   - $F(n) = 2F(n-1) + F(n-2)$ for $n \geq 3$

   Prove that $F(n) \geq 3 \times 2^{n-1}$ all $n \geq 1$.
2. Prove that every tree with at least two vertices has at least two nodes of degree 1.
3. Prove that every tree with $n \geq 1$ vertices has exactly $n - 1$ edges.
4. Prove that every tree can be properly vertex colored with 2 colors.
5. Suppose that $G = (V, E)$ satisfies $degree(v) \leq D$ for every vertex $v \in V$. Prove G can be properly vertex colored with $D + 1$ colors.
6. Suppose $S_0 = \{0\}$ and $S_n = S_{n-1} \cup \{n^2\}$ for $n \geq 1$. Prove that $S_n = \{x^2 : 0 \leq x \leq n, x \in \mathbb{N}\}$ for all $n \in \mathbb{N}$.

6 Proof by contradiction

Problems:

1. Prove that the set of primes is infinite.
2. Prove that there is no largest real number less than 1.
3. Prove that the cube root of 15 is irrational.
4. Prove that $\mathbb{P}(\mathbb{N})$ is uncountable.
5. Prove that the irrationals are uncountable.
6. Prove (by contradiction) every statement you were asked to prove by induction (see Section 5).

7 $\mathcal{P}$, $\mathcal{NP}$, $\mathcal{NP}$-hard, $\mathcal{NP}$-complete, and reductions between problems

Basic knowledge:

1. Know the definitions of $\mathcal{P}$, $\mathcal{NP}$, $\mathcal{NP}$-hard, and $\mathcal{NP}$-complete.
2. Know the properties a Karp reduction has to satisfy.
3. Understand the differences between decision, optimization, and construction problems, and how an algorithm to solve one of the problems can be used to solve the others.

Problems:

1. Suppose you have an algorithm $\mathcal{X}$ that can solve the decision problem for CLIQUE. Show how to use $\mathcal{X}$ to find the largest clique in a graph. You are allowed to use $\mathcal{X}$ a polynomial number of times, and otherwise you can do a polynomial amount of work.
2. Suppose you have an algorithm $\mathcal{X}$ that can solve the decision problem for HAMILTONIAN PATH. Show how to use $\mathcal{X}$ to find the Hamiltonian path (if it exists) in a graph. You are allowed to use $\mathcal{X}$ a polynomial number of times, and otherwise you can do a polynomial amount of work.
3. Suppose you have an algorithm $\mathcal{A}$ and $\mathcal{A}(G)$ returns the size of the largest clique in the graph G.
   - Describe an algorithm $\mathcal{B}$ that can find the size of the largest independent set in a graph, and that satisfies:
     - $\mathcal{B}$ can call $\mathcal{A}$ at most once (but on any input graph it likes)
     - For all other steps, $\mathcal{B}$ runs in polynomial time

     Show all the steps, so that given an input graph G, the result of running $\mathcal{B}(G)$ is the size of the largest independent set in G.
   - Suppose that $\mathcal{A}$ runs in polynomial time. Could you answer the question "Does $\mathcal{P} = \mathcal{NP}$?"
   - Now show an algorithm $\mathcal{C}$ to find a smallest vertex cover in a graph, where
     - $\mathcal{C}$ can call $\mathcal{B}$ once (but on any graph)
     - For all other steps, $\mathcal{C}$ runs in polynomial time
4. (*) The 3-COLORABILITY decision problem takes as input a graph G and return YES if and only if the graph can be vertex-colored with 3 colors (similarly for 4-COLORABILITY). Give the Karp reduction that maps 3-colorability to 4-colorability. Prove that it has the required properties.
5. (*) The 3-CLIQUE problem is a decision problem that takes as input a graph G and returns YES if and only if G has a clique of size 3. Similarly for the 4-CLIQUE problem. Give the Karp reduction that maps 3-CLIQUE to 4-CLIQUE. Prove that it has the required properties.

6. (*) Be able to show that VERTEX COVER is NP-complete, using the fact that INDEPENDENT SET is NP-complete.

7. (*) Be able to show that CLIQUE is NP-complete using the fact that INDEPENDENT SET is NP-complete.

8. (*) The CLIQUE decision problem takes as input a graph G and an integer k, and returns *Yes* if and only if G has a clique of size k. Similarly the INDEPENDENT SET decision problem takes as input a graph G and an integer k and returns *Yes* if and ony if G has an independent set of size k. Give a Karp reduction from CLIQUE to INDEPENDENT SET, and prove that it has the required properties.