

Preparing for the CS 173 (B) Midterm

To prepare for the midterm, make sure you can do all problems that appeared in any prior homework, examlet, or reading quiz, and all the material that was taught in class (even if it did not appear in any examlet, homework, or reading quiz).

In addition, make sure you can do the following set of problems; the midterm is likely to include very similar problems to these.

1. Prove by contradiction that when p and q are distinct primes, that $(pq)^{\frac{1}{2}}$ is not a rational number.
2. Prove by contradiction that there is an infinite number of finite subsets of the positive integers.
3. Prove that the set $A = \{x \in \mathbb{Z}^+ : x \geq 3\}$ is infinite by providing a bijection F between A and a proper subset of A . Note: you need to prove that F is a bijection.
4. Consider the function $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined by
 - $F(1, m) = F(m, 1) = m, \forall m \in \mathbb{Z}^+$
 - $F(m, n) = \frac{F(m-1, n) + F(m, n-1) + m + n}{2}, \forall (m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $m > 1$ and $n > 1$.

Prove $F(m, n) = mn$ for all $(m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ by induction.

5. Let R denote the real numbers, and let function $F_n : R \rightarrow R$ be defined recursively for $n = 0, 1, \dots$, as follows:

- $F_0(x) = x, \forall x \in R$
- $F_n(x) = 2F_{n-1}(x), \forall x \in R, \text{ when } n > 0.$

Find a closed form formula for $F_n(x)$ and prove your formula correct by induction on n .

6. Let the sets A_1, A_2, \dots be defined by

- $A_1 = \{3\}$
- $A_n = \{x : \exists y \in A_{n-1} \text{ such that } x = y + 1\}, \text{ for } n > 1.$

Figure out what A_2 and A_3 are. Then, find a closed form solution for A_n and prove it correct by induction on n .

7. Let $A_{i,j}$ be a set that is defined recursively, by $A_{0,i} = \{2i\}, A_{i,0} = \{3i\}$, and $A_{i,j} = A_{i-1,j} \cup A_{i,j-1}$ if i, j are both positive integers.

- Determine $A_{i,j}$ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$.
- Find a closed form solution for $A_{i,j}$.
- Prove your closed form solution correct by induction.

8. Consider the following sets:

- $A = \{S \in \mathcal{P}(Z) : 1 \notin S\}$,
- $B = \{S \in \mathcal{P}(Z) : 1 \in S\}$,
- $C = \{S \in \mathcal{P}(Z) : 2 \notin S\}$, and
- $D = \mathcal{P}(Z)$.

(a) Draw the Venn Diagram for these four sets, and determine whether each of the areas is empty or non-empty. For each non-empty area, provide an example of an element to prove it's non-empty. For each of the empty areas, prove that it is empty.

(b) Is $2 \in A$?

(c) Is $A \cap B = \emptyset$?

9. Consider the sets

- $A = \{f : Z^+ \rightarrow Z^+\}$
- $B = \{f \in A : f(Z^+) \neq Z^+\}$
- $C = \{f \in A : \forall z \in Z^+, \exists z' \in Z^+ \text{ s.t. } f(z') = z\}$.

Draw the Venn Diagram for these three sets, and prove whether each area is empty or not.

10. Consider the following formula:

$$(A \Rightarrow B) \wedge (B \Rightarrow \neg C) \wedge (B \Rightarrow C)$$

- Is this formula a tautology, satisfiable but not a tautology, or not satisfiable? (Prove your answer)
- Rewrite the formula in 2CNF form.

11. Rewrite the following 2CNF expression so that it is a conjunction of clauses of the form $X \Rightarrow Y$:

$$(\neg A \vee \neg B) \wedge (B \vee C) \wedge (\neg C \vee A)$$

12. Consider the 2CNF formula

$$(\neg A \vee \neg B) \wedge (B \vee C) \wedge (\neg C \vee A)$$

- Construct the directed graph for this formula
- Does the directed graph have a directed cycle containing both X and $\neg X$ for some variable X ?
- Is the formula satisfiable?
- Find a satisfying assignment if the formula is satisfiable.

13. Consider the 2CNF formula given by

$$(\neg A \vee B) \wedge (\neg B \vee \neg C) \wedge (\neg B \vee C)$$

- Construct the directed graph for this formula
- Is the formula satisfiable?
- Find a satisfying assignment if the formula is satisfiable
- Simplify the formula so that it has no parentheses and no implications (\Rightarrow)

14. Let A be the set of non-empty subsets of Z^+ and let R be the binary relation on A defined by $(X, Y) \in R$ if and only if $\sum_{x \in X} x = \sum_{y \in Y} y$.

- (a) Give an example of an element of R , and a pair (X, Y) of subsets of Z^+ such that $(X, Y) \notin R$.
- (b) Prove or disprove: R is reflexive, R is irreflexive, R is symmetric, R is anti-symmetric, R is transitive.
- (c) Think about what would change if you had replaced $\sum_{x \in X} x = \sum_{y \in Y} y$ by $\sum_{x \in X} x \leq \sum_{y \in Y} y$.

15. Let A be the set of functions $F : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$. Let R be the binary relation on A defined by $(f, g) \in R$ if and only if $Image(f) = Image(g)$.

- (a) Give an example of an element of R , and a pair (f, g) of functions such that $(f, g) \notin R$.
- (b) Prove or disprove: R is reflexive, R is irreflexive, R is symmetric, R is anti-symmetric, R is transitive.

Harder questions The remaining questions are harder ones, and I am not going to put problems like these on the midterm. However, problems like these could show up on the later homeworks, or on the final. And anyway, some of you might enjoy thinking about them.

1. Let X be the set of finite non-empty subsets of Z^+ and let Y be the set of infinite subsets of Z^+ . Draw the Venn Diagram of the sets X, Y , and $\mathcal{P}(Z^+)$. Prove whether each area is empty or non-empty (for the non-empty areas, give examples of elements in the area).
2. For the same definitions of X and Y in the previous problem:
 - (a) Prove that X is infinite
 - (b) Prove that Y is infinite
 - (c) Hard problem: find a 1 – 1 function $F : X \rightarrow Y$, and prove it is 1-1.
 - (d) Hard problem: prove that X is countably infinite

- (e) Hard problem: prove that Y is uncountable
 - (f) Hard problem: prove that there is no bijection between Y and X
3. Hard problem: Think about the graph algorithm for solving 2SAT. In the class, I showed you how to compute the directed graph from a 2CNF formula, and I asserted that the 2CNF formula is satisfiable if and only if the graph had no directed cycle containing both X and $\neg X$ for any variable X . See if you can prove this. In particular, show how to find a satisfying assignment if the directed graph does not contain both X and $\neg X$ for any variable X . Note: you need to think carefully about what the satisfying assignment would contain if the graph has a directed path from X to $\neg X$.
 4. Let X_n denote the set of subsets of Z^+ that have at most n elements (i.e., $X_n = \{A \subseteq Z^+ : |A| \leq n\}$). What can you say about $X = \bigcup_n X_n$? Is it equal to $\mathcal{P}(Z^+)$? Is X_1 finite? Is there any element of X that is infinite? Now let Y_n denote the set of subsets of $\{1, 2, \dots, n\}$, and let $Y = \bigcup_n Y_n$. Is $Y = \mathcal{P}(Z^+)$? Is there any element of Y that is infinite? Draw the Venn Diagram for X and Y , and show whether the areas are empty or non-empty.