Honors add-on, Homework #2 solution

In the two-person game described for this homework, the players take turns taking rocks off one or both piles, but they must take at least one rock off and cannot take more than one rock off of either pile.

You are asked to prove that the first player has a winning strategy if and only if at least one pile has an odd number of rocks. Here is the proof.

Theorem: Let pile 1 have K rocks and pile 2 have L rocks. Then the first player has a winning strategy if and only if K is odd or L is odd.

Proof: The proof is by induction on $K + L$.

The inductive hypothesis is that there exists some positive integer N such that for all non-negative integers K, L with $K + L \leq N$, the first player has a winning strategy if and only if K is odd or L is odd.

The base case is $N = 1$. When $N = 1$, then $K = 1$ and $L = 0$, or vice-versa. In both cases, the first player has a winning strategy (take the single rock off its pile). Furthermore, at least one of the two piles has an odd number of rocks. Hence the base case is established.

Now consider two piles of rocks with K rocks on pile 1 and L rocks on pile 2, where $K + L = N + 1$. We will show that

the first player has a winning strategy $\Leftrightarrow$ at least one of K or L is odd.

Thus we need to show (a) that if the first player has a winning strategy, then at least one of K or L is odd, and (b) if one of K or L is odd, then the first player has a winning strategy.

$\Rightarrow$. Suppose the first player (Alice) has a winning strategy given starting position (K, L). After Alice moves, the new position is one of the following: $(K-1, L), (K, L-1)$ or $(K-1, L-1)$. Since Alice has a winning strategy, whatever she did will by definition not create a position where Bob (her opponent) has a winning strategy. Now note that the total number of rocks has decreased, and so we can apply the indutive hypothesis. Since Bob cannot have a winning strategy, it must be that **the number of rocks on both piles is even**. Hence, whatever Alice did, she created a condition where both piles have an even number of rocks. Since she could only take one rock off of each pile, and was required to take off at least one rock, it follows that she could not begin with a condition that had no pile with an odd number of rocks. Hence, if Alice has a winning strategy, then at least one of her piles has an odd number of rocks.

$\Leftarrow$. If at least one of K or L is odd, then Alice can remove rocks from the piles to make the number of rocks on both piles even. The resultant condition has fewer rocks in total, and so the inductive hypothesis applies. Hence, Bob is presented with a condition in which both piles have an even number of rocks, and by the inductive hypothesis Bob does not have a winning strategy.

Since N was arbitrary, this proof establishes that the property is true for all numbers n of rocks distributed between the two piles.