CS 173 (A) Fall 2018
Solutions (and grading rubric)
FINAL Examination
December 15, 2018

NAME (PRINTED):

Acceptance of Honor code (Signed):

NETID (e.g. hpotter23, not 314159265):

IMPORTANT: Circle your discussion:

ADA ADB ADC ADD ADE ADF ADG ADH ADI ADJ

IMPORTANT: Circle your T.A.:

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Problem	1	2	3	4	5	6	7	8	9	10
Possible	10	10	18	10	11	7	8	9	10	7
Score										

Total out of 100 points

Instructions: only show your work for Problem 1; for everything else, just show your answers.

Problem 1 (10 points)

Recall that weak induction is sometimes sufficient, but you can always turn any weak induction proof into a strong induction proof. Take a look at the attempted proof below, and see if it's valid (if not, say why not). Also, see how to change it into a strong induction proof. The lines of the proof are numbered so you can comment on the proof in detail.

Let $f : \mathbb{N} \to \mathbb{N}$ be defined recursively by:

- $f(0) = f(1) = 10$
- $f(n) = 100 \times f(n-2) + f(n-1)$ for $n \geq 2$

We wish to prove that $f(n) \geq 10^n$ for all $n \in \mathbb{N}$.

1. We note that $f(0) = 10$ and $10^0 = 1$. Hence $f(n) \geq 10^n$ for $n = 0$.
2. We note that $f(1) = 10$ and $10^1 = 10$. Hence $f(n) \geq 10^n$ for $n = 1$.
3. Let $P(n)$ be the statement that $f(n) \geq 10^n$.
4. Hence we have already shown that $P(0)$ and $P(1)$ are true.
5. Our Inductive Hypothesis is that $P(N)$ is true for some arbitrary integer $N \geq 1$.
6. Hence, $f(N) \geq 10^N$.
7. We wish to show that $P(N) \to P(N+1)$.
8. Since $N \geq 1$ it follows that $N + 1 \geq 2$.
9. Hence, by the recursive definition for f, we see that $f(N+1) = 100 \times f(N-1) + f(N)$.
10. Therefore, by the Inductive Hypothesis, $f(N+1) \geq 100 \times 10^{N-1} + 10^N = 10^{N+1} + 10^N \geq 10^{N+1}$.
11. Hence $P(N+1)$ is true
12. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$ (i.e., $f(n) \geq 10^n$ for all $n \in \mathbb{N}$).
13. *q.e.d.*

(5 pts) Is this a valid proof by induction? (Explain) Answer is No, and the reason is that with weak induction there is no way to say anything about $f(N-1)$. Hence Line 10 fails.

(5 pts) Modify line 5 so that you have a strong inductive hypothesis. Answer: The inductive hypothesis is that $P(i)$ is true for all i between 0 and N for some arbitrary integer $N \geq 1$.

Problem 2 (10 pts)

Instructions to grader: No partial credit on any part of this problem.

1. (5 pts) For each of the following, determine if it is a tautology, satisfiable but not a tautology, or always false. Write "T" if tautology, "S" if satisfiable but not a tautology, and "F" if always false

   Instructions to grader: Give 1 point per problem if the correct answer is shown (T, S, or F, or written out as Tautology, Satisfiable, or False).

   (a) $(P \vee Q) \rightarrow (P \wedge \neg Q)$
   Satisfiable

   (b) $(A \wedge B) \vee (\neg A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge \neg B)$
   Tautology

   (c) $(a \rightarrow \neg b) \wedge (\neg b \vee a) \wedge b$
   False

   (d) $(P \rightarrow \neg Q) \wedge (\neg Q \rightarrow \neg P)$
   Satisfiable

   (e) $(P \rightarrow \neg Q) \wedge (\neg Q \rightarrow P)$
   Satisfiable

2. (5 pts) Give a satisfying assignment for $(\neg Q \rightarrow Q) \wedge (Q \rightarrow P)$
   Q: **T**, P: **T**.

Problem 3 (18 points)

The following questions are for the graph $G = (V, E)$ given on the last page of the exam (you may wish to detach that page so you can look at it as you answer these questions.

Instructions to grader: Partial credit for some of these questions, as indicated.

1. (1 pt) Does G have an Eulerian circuit? (yes or no) Answer: NO.
2. (1 pt) Can G be 2-colored? (yes or no) NO.
3. (2 pts) Write down the vertices in a minimum dominating set for G

   2 points for a dominating set of size 3 (e.g., $\{a, b, j\}$). 1 point for a dominating set of size 4. Anything else, give no points.
4. (2 pts) Write down the vertices in a minimum vertex cover for G

   2 points for a vertex cover with 5 vertices (e.g., $\{a, b, f, g, i\}$). 1 point for a vertex cover of size 6. All other answers, no points.
5. (2 pts) Write down the edges in a maximum matching for G ANSWER: any matching with 5 edges.

   2 pts for a matching with 5 edges. 1 pt for a matching with 4 edges. 0 pts all other answers.

6. (10 pts) (This problem refers to the graph on the last page of your exam.) Consider the following properties that may or may not be true of any given set $A \subseteq V$

- $P_1 : \forall \{a, b\} \subseteq A, (a, b) \in E$
  Note: this means A is a clique
- $P_2 : \forall \{a, b\} \subseteq A, (a, b) \notin E$
  Note: this means A is an independent set
- $P_3 : \forall b \in V \setminus A \ \exists a \in A \ s.t. \ (a, b) \in E$
  Note: this means A is a dominating set
- $P_4 : \exists a \in A \ \exists b \in V \setminus A \ s.t. \ (a, b) \in E$
  Note: this is only false when A is the union of components and is not all of V

For each of the following sets A_i, list all the properties from the above list that it satisfies.

Instructions to graders. Partial credit is permitted, as indicated for each question.

- $A_1 = V$
  Properties this set satisfies: P_3
  1 point for a perfect answer; any other answer, 0 pts.
- $A_2 = \{a, h\}$
  Properties this set satisfies: P_1, P_4
  2 points for a perfect answer. If they give one of the two properties and nothing else, give 1 point. All other answers: no points.
- $A_3 = \{c, b, f, g, i\}$
  Properties this set satisfies: P_2, P_3, P_4
  3 points for a perfect answer. Give one point for each appearance of a correct property (P_2, P_3, P_4) and take off a point if they give P_1.
- $A_4 = \emptyset$
  Properties this set satisfies: P_1, P_2
  1 point for a perfect answer (everything else 0 points).
- $A_5 = \{d\}$
  Properties this set satisfies: P_1, P_2, P_4
  3 points for a perfect answer. Give 1 point for each correct property, and take off 1 point if they include P_3.

Problem 4 (10 points)

Instructions to graders: No partial credit on this page, but partial credit on the next page is possible, as noted.

1. (4 pts) Give the formulas for each of the following (Instructions: 1 point if correct, 0 otherwise)
   (a) $C(n,k)$ where $n \geq k \geq 1$
   The correct answer is $\frac{n!}{k!(n-k)!}$.
   (b) $|\mathcal{P}(X)|$, where X is a set with n elements and $\mathcal{P}(X)$ denotes the power set of X.
   The correct answer is 2^n.
   (c) The number of ways you can order all the elements of a set S that contains n elements
   The correct answer is $n!$
   (d) The number of functions $f : S \to A$ where S has n elements and A has k elements
   The correct answer is k^n.

2. (2 pts) How many ways can you divide a set of $2n$ people into two teams with n people each? Note that the answer when $n = 1$ should be 1.
   The correct answer is $C(2n, n)/2$. They might use other notation for this, or this could be written out as a formula, which is also fine. If they write $C(2n, n)$, give 1 point.

3. (2 pts) How many ways can you divide a set of $2n$ people into two teams with n people each, if Henry and Sally have to be in the same team?
   The correct answer is $C(2n - 2, n - 2)$, but this is the same as $C(2n - 2, n)$. For example, pick which n-2 people go with Harry and Sally. This could be written out as a formula, which is also fine. The only partial credit is if they multiply this or divide this by 2 - in that case give one point.

4. (2 pts) How many ways can you divide a set of $2n$ people into two teams with n people each, if Henry and Sally refuse to be in the same team? (For this problem, assume that $n > 1$, so that a feasible solution always exists.)
   The correct answer is $C(2n-2, n-1)$ – i.e., pick who goes with Harry, but don't allow yourself to pick Sally. Another answer is the difference between the answers to the first two questions. The only partial credit is if they multiply this or divide this by 2 - in that case give one point.

Problem 5 (11 points)

Instructions: no partial credit for this problem - both pages.

1. (6 pts) Which of the following are valid definitions of the statement that f is $O(g)$, where f and g are both functions from the real numbers to the real numbers? Check the box after the definition if it's valid.

   (a) $\exists C > 0$ such that $|f(n)| \leq C|g(n)|$ $\forall n > 0$

   (b) $\exists C > 0$ such that $|f(n)| \geq C|g(n)|$ $\forall n > 0$

   (c) $\exists C > 0, N > 0$ such that $|f(n)| \leq C|g(n)|$ $\forall n > N$ VALID!

   (d) $\exists C > 0, N > 0$ such that $|f(n)| \geq C|g(n)|$ $\forall n > N$

   (e) $\exists C > 0$ such that $C|f(n)| \leq |g(n)|$ for an infinite number of values of n

   (f) $\exists C > 0$ such that $C|f(n)| \leq |g(n)|$ for all but a finite number of values of n VALID!

2. (5 pts) For each pair of functions f and g given below (with both mapping natural numbers to natural numbers) say whether f is $O(g)$ and whether g is $O(f)$. Note that it is possible for both to be Big-O of each other or for neither to be Big-O of each other. Put your answers for each such question in the table *below* the list of pairs of functions.

   (a) $f(n) = 1$ and $g(n) = 0$
   (b) $f(n) = n^2$, $g(n) = f(n) - 3$
   (c) $f(n) = 5n$, $g(n) = f(n/2)$
   (d) $f(n) = n^4 + 5000$, $g(n) = n^3 - 5000$
   (e) $f(n) = 1$ if n is odd and $f(n) = n$ if n is even, and $g(n) = 2n$ if n is divisible by 6 and $g(n) = 5$ otherwise

   Instructions to graders: half a point for each correct entry in the box below.

Table 1: Instructions: Put a checkmark or "x" if it's true

	f is $O(g)$	g is $O(f)$
(a)		x
(b)	x	x
(c)	x	x
(d)		x
(e)		x

Problem 6 (7 points)

Instructions to graders: partial credit, as indicated.

1. (3 pts) Suppose you have two oracles for construction problems, with Oracle A being for Maximum Clique and an Oracle B for Maximum Independent Set. You want to solve Minimum Vertex Cover. You have a finite simple graph $G = (V, E)$. You give the graph to the two oracles and you get the following answers:

   - Oracle A returns set Q
   - Oracle B returns set R

   What do you return as a Minimum Vertex Cover for G?

   Answer is $V \setminus R$.

   They might write this differently, such as $V - R, V^c, \bar{V}$, etc. Or they might write it out in English, such as "return all the vertices not in R". All these get full points. No partial credit.

No partial credit for this question. Each correct answer gets 1 point.

2. (4 pts) Suppose $G = (V, E)$ is a graph and you form H by adding three vertices $\{a, b, c\}$ (none of which is already in G), make $\{a, b, c\}$ into a clique (i.e., you add edges $(a, b), (a, c)$, and (b, c)), and you also make a, b, and c adjacent to all the vertices in V. Using set-theoretic notation, $H = (V', E')$, where

   - $V' = V \cup \{a, b, c\}$
   - $E' = E \cup \{(x, y) | y \in \{a, b, c\}, x \in V\} \cup \{(a, b), (b, c), (a, c)\}$

   Suppose now you find out that

   - H has a maximum clique size of 17
   - H can be vertex colored with 20 colors but not with fewer than 20 colors
   - and the maximum independent set size in H is 10.

   Answer the following questions:

   (a) The maximum clique size of G is:
   14

   (b) The maximum independent set size of G is:
   10

   (c) The minimum number of colors needed to vertex-color G is:
   17

   (d) The minimum dominating set size of H is:
   1

Problem 7 (8 pts)

No partial credit for this question.

1. Let $f_n : \mathbb{N} \to \mathbb{N}$, for $n \geq 1$, be functions that are defined recursively by:

   - $f_1(x) = x$ for all $x \in \mathbb{N}$
   - If $n > 1$ then $f_n(x) = x + f_{n-1}(x)$ for all $x \in \mathbb{N}$

   Answer the following questions:

   - (1 pt) What is $f_1(5)$?
     5
   - (1 pt) What is $f_2(5)$?
     10
   - (1 pt) What is $f_3(5)$?
     15
   - (1 pt) What is $f_4(5)$?
     20
   - (1 pt) What is $f_5(5)$?
     25
   - (3 points) What is a closed form solution for $f_n(x)$? (Express this as a function of n and x that has no recursive calculation.)
     nx

Problem 8 (9 points).

For each of the following problems, consider whether it is established to be in $\mathcal{NP}, \mathcal{P}$, $\mathcal{NP}$-hard, and $\mathcal{NP}$-complete. Also answer whether it is known to be solvable in polynomial time. (You might as well assume for this problem that $\mathcal{P} \neq \mathcal{NP}$.)

- P_1 is the problem that takes as input a graph G and answers True if the graph G can be 3-colored (and otherwise answers False)
  This problem is NP-complete. Hence, in NP and NP-hard.
- P_2 is the problem that takes as input a graph G and answers True if the graph G can be 2-colored (and otherwise answers False)
  This problem is solvable in polynomial time and in P and in NP.
- P_3 is the problem that takes as input a graph G and returns a maximum clique in G
  P_3 is NP-hard but not in NP. Hence not NP-complete.
- P_4 is the problem that takes as input a graph G and answers True if the graph G has an Eulerian Circuit (and otherwise answers False)
  This problem is solvable in polynomial time and in P and in NP.
- P_5 is the problem that takes as input a graph G and answers True if the graph G has a Hamiltonian Path (and otherwise answers False)
  P_5 is NP-complete. Hence, in NP and NP-hard.
- P_6 is the problem that takes as input a CNF formula X and answers True if X is satisfiable (and otherwise answers False)
  P_6 is NP-complete. Hence, in NP and NP-hard.
- P_7 is the problem that takes as input a 2CNF formula X and answers True if X is satisfiable (and otherwise answers False)
  This problem is solvable in polynomial time and in P and in NP.
- P_8 is the problem that takes as input a graph G that is required to be acyclic and connected (i.e., a tree) and answers True if G can be 3-colored (and otherwise answers False)
  This problem is solvable in polynomial time and in P and in NP.
- P_9 is the problem that takes as input a wheel graph G and returns the size of the maximum clique in G
  This problem is solvable in polynomial time but is not in NP and so not in P.

Instructions to grader: This problem is worth 9 points, so each incorrectly handled box results in 2/10 of a point taken off from 9.

Table 2: Solutions (put a check or "x" if it is True)

	in $\mathcal{NP}$?	in $\mathcal{P}$?	$\mathcal{NP}$-hard?	solvable in polynomial time?	$\mathcal{NP}$-complete?
P_1	x		x		x
P_2	x	x		x	
P_3			x		
P_4	x	x		x	
P_5	x		x		x
P_6	x		x		x
P_7	x	x		x	
P_8	x	x		x	
P_9				x	

Problem 9 (10 pts)

Instructions: For this page, 2 points each problem, no partial credit.

1. (6 pts) Consider the edge-weighted graph $G = (V, E)$ with

   - $V = \{v_1, v_2, \ldots, v_{10}\}$
   - $E = \{(v_i, v_j) | 1 \le i < j \le 10\}$ (i.e, G is a clique)
   - $w(v_i, v_j) = ij$ for all i, j with $1 \le i < j \le 10$

   Recall the Floyd-Warshall algorithm (i.e., the dynamic programming algorithm presented in class for all-pairs shortest paths).

   (a) What is $M[4, 5, 0]$? 20

   (b) What is $M[4, 5, 1]$? 9

   (c) What is $M[4, 5, 2]$? 9

Instructions: For this page, partial credit as indicated.

2. Consider the sequence A $= 5, 3, 1, 4, 8, 9, 2, 1, -1, 6, 10, 14, 11, 12, 15, 7$. Recall the dynamic programming algorithm presented in class for Longest Increasing Subsequence.

   - (3 pts) Write out the matrix $Q[1..16]$
     The correct answer is

     [1,1,1,2,3,4,2,1,1,3,5,6,6,7,8,4]

     If they get this exactly correct, give 3 points. If their first mistake is in positions 1...6 (given in red, above), give 1 point. If their first mistake is after position 6, give 2 points.

   - (1 pt) Write down a longest increasing subsequence for A
     There are two valid answers, that differ only in the first entry, which can be 1 or 3. The two answers are:

     $$1, 4, 8, 9, 10, 11, 12, 15$$

     $$3, 4, 8, 9, 10, 11, 12, 15$$

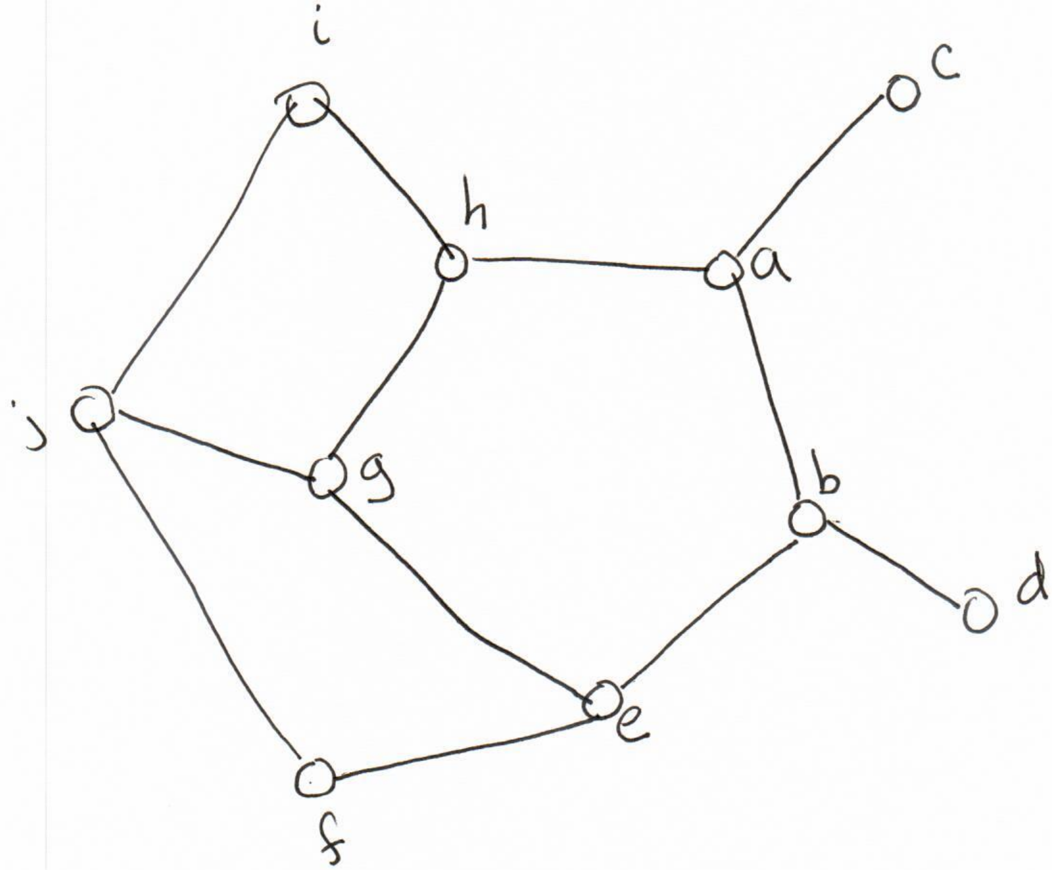
     They get full points if they give one of these, and no points otherwise.

Problem 10 (7 pts)

Instructions: 1 point per problem, no partial credit.

For each set, write **F** if it is finite, **C** if it is countably infinite, and **U** if it is uncountable.

1. $\mathbb{N}$
   C - countably infinite
2. $\mathbb{Z} \times \mathbb{N}$
   C - countably infinite
3. $\{f : \{0, 1, 2, 3\} \to \{1, 2, 3, 4, 5\}\}$
   F - finite
4. $\{f : \mathbb{N} \to \{1, 2\}\}$
   U - uncountable
5. $\mathbb{R}$
   U - uncountable
6. The set of all finite length strings over $\{0, 1\}$
   C - countably infinite
7. The set of all infinite length strings over $\{0, 1\}$
   U - uncountable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