CS 173, Fall 2015
Examlet 8, B Lecture
December 3, 2015
Solutions

Problem 1: (10 pts) Prove that every connected acyclic simple graph G with $n \geq 2$ vertices has at least two vertices that have degree 1.

Solution: Let G be an arbitrary connected acyclic simple graph with at least two vertices. Let P be a longest path *in G (where the length of the path is the number of edges in the path), with $P = v_1, v_2, \ldots, v_k$. Since G has two vertices, $k \geq 2$ (i.e., the path has at least one edge in it). We will prove by contradiction that v_1 and v_k are both leaves. Suppose that v_1 is not a leaf. Hence v_1 has at least two neighbors, and so some vertex $w \neq v_2$ is a neighbor of v_1. If $w \neq v_i, 3 \leq i \leq k$, then we can create a path $P' = w, v_1, v_2, \ldots, v_k$ that is longer than P, which is a contradiction. Therefore w must be one of the vertices in P besides v_2. But then $w = v_i$ for some $i \geq 3$. Therefore $w, v_1, v_2, \ldots, v_i, w$ is a cycle, which contradicts the assumption that G is acyclic. Hence v_1 has degree 1. The same argument shows that v_k has degree 1. Hence G has at least two nodes of degree 1.*

Problem 2: (10 pts) Prove by contradiction that $\mathbb{P}(\mathbb{Z}^+)$ is uncountable.

Solution: Suppose that $\mathbb{P}(\mathbb{Z}^+)$ is countable. Then we can write it as $X_1, X_2, \ldots$. Consider the infinite matrix M where the first row of M defines X_1, the second row defines X_2, etc. Furthermore, we set $M[i,j] = 1$ if and only if $j \in X_i$, and otherwise $M[i,j] = 0$. Consider the set $Y = \{j : j \notin X_j\}$; i.e., $Y = \{j : M[j,j] = 0\}$. This set Y cannot be equal to any X_j since $j \in X_j$ if and only if $j \notin Y$. Hence Y is not in the enumeration of $\mathbb{P}(\mathbb{Z}^+)$, contradicting our hypothesis. Therefore $\mathbb{P}(\mathbb{Z}^+)$ is not countable.

Table 1: Adjacency matrix for problem 3

	a	b	c	d	e	x
a	0	1	0	0	1	1
b	1	0	1	0	0	1
c	0	1	0	1	0	1
d	0	0	1	0	1	1
e	1	0	0	1	0	1
x	1	1	1	1	1	0

Problem 3: (10 pts) Consider the graph described by the adjacency matrix in Table 1.

1. Draw the graph (2 pts)
2. Draw the Breadth-First-Search (BFS) tree starting at vertex a (2 pts)
3. Find a minimum dominating set (2 pts)
4. Find a maximum matching (2 pts)
5. Find a minimum vertex coloring for the graph (2 pts)

Solution: This is the wheel graph with 5 vertices around the outside, and one vertex in the middle (x).

The BFS tree starting at a will have edges $(a, b), (a, e), (a, x), (b, c), (e, d)$.

The minimum dominating set is $\{x\}$.

There are five different maximum matchings, and here are two: $\{(a, b), (c, d), (x, e)\}$ *and* $\{(b, c), (d, e), (a, x)\}$.

A minimum vertex coloring requires four colors. We know this because x has to get a color different from every other vertex, and the 5-cycle a-b-c-d-e-a needs three colors. Here is one of the minimum vertex colorings: $f(x) = 1, f(a) = f(c) = 2, f(b) = f(d) = 3, f(e) = 4$.