

Solution to Examlet #2

CS 173, Lecture B

Let S_n , $n \in \mathbb{Z}^+$, be defined by $S_1 = \{1\}$, and $S_n = S_{n-1} \cup \{n\}$ if $n > 1$. Prove that $S_n = \{i : 1 \leq i \leq n, i \in \mathbb{Z}\}$ for all integers $n \geq 1$.

Proof by contradiction. We begin with the proof by contradiction. Let $P(n)$ be the statement $S_n = \{i : 1 \leq i \leq n, i \in \mathbb{Z}\}$. Note that $P(1)$ is true if and only if $\{1\} = \{i : 1 \leq i \leq 1, i \in \mathbb{Z}\}$. Hence, $P(1)$ is true.

If it is not the case that $P(n)$ is true for all positive integers n , then there is a smallest positive integer N for which $P(N)$ is false. Note that $N > 1$. Hence, $N - 1 \geq 1$, and therefore $P(N - 1)$ is true.

Since $N > 1$, by definition, $S_N = S_{N-1} \cup \{N\}$.

Since $P(N - 1)$ is true, $S_{N-1} = \{i : 1 \leq i \leq N - 1, i \in \mathbb{Z}\}$. Hence,

$$\begin{aligned} S_N &= \{i : 1 \leq i \leq N - 1, i \in \mathbb{Z}\} \cup \{N\} \\ &= \{i : 1 \leq i \leq N, i \in \mathbb{Z}\} \end{aligned}$$

In other words, $P(N)$ is true, contradicting our assumption. Therefore, $P(n)$ must be true for all $n \in \mathbb{Z}^+$.

Proof by induction. Again we let $P(n)$ be the statement

$$S_n = \{i : 1 \leq i \leq n, i \in \mathbb{Z}\}$$

Note that $P(1)$ is true if and only if $\{1\} = \{i : 1 \leq i \leq 1, i \in \mathbb{Z}\}$. Hence, $P(1)$ is true.

Our inductive hypothesis is that there is some positive integer K such that for all $n, 1 \leq n \leq K$, $P(n)$ is true. Since we have shown that $P(1)$ is true, it follows that $K \geq 1$. We now want to prove that $P(K + 1)$ is true.

So consider S_{K+1} . Since $K \geq 1$, it follows that $K + 1 \geq 2$, and so by definition $S_{K+1} = S_K \cup \{K+1\}$.

By the inductive hypothesis, $S_K = \{1, 2, \dots, K\}$. Hence,

$$S_{K+1} = \{1, 2, \dots, K\} \cup \{K + 1\} = \{1, 2, \dots, K + 1\}.$$

Hence, $P(K+1)$ is true.

Since K was arbitrary, $P(n)$ is true for all positive integers n .