CS173
More about Strong Induction Proofs

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Today

We will cover

- Review of weak and strong induction
- Practicing strong induction

Weak Induction vs. Strong Induction

- Weak Induction asserts a property $P(n)$ for one value of n (however arbitrary)
- Strong Induction asserts a property $P(k)$ is true for all values of k starting with a base case n_0 and up to some final value n.
- The same formulation for $P(n)$ is usually good - the difference is whether you assume it is true for just one value of n or an entire range of values.

Sometimes Strong Induction is needed.

Recurrence relations

Recurrence relations are generally functions defined recursively:

1. $g(1) = 3$ and $g(n) = 3 + g(n-1)$ for $n \geq 2$
2. $f(1) = f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$.
3. $h(1) = 1, h(2) = 2$ and $h(n) = h(1) + h(2) + \ldots + h(n-1)$ if $n \geq 3$

Note that

- $g(n)$ only depends on $g(n-1)$
- $f(n)$ depends on $f(n-1)$ and $f(n-2)$, and
- $h(n)$ depends on $h(1), h(2), \ldots, h(n-1)$

Hence you *must* use strong induction for anything you want to prove about $f(n)$ or $h(n)$ but you *could* have used weak induction for $g(n)$.

Strong induction is always valid, so practice using it.

Strong Induction

Points:

- Helpful to always state what you want to prove as a boolean statement, $P(n)$, that depends on a parameter n
- Explicitly check the base cases
- Explicitly write down your Inductive Hypothesis: For example, "Our Inductive Hypothesis is that $P(1) \wedge P(2) \ldots \wedge P(N)$ is true for some arbitrary $N \geq n_1$" (where n_1 is the largest base case you checked)
- Make sure your proof uses the information in your problem (e.g., if you are given a recursively defined function, use the its recursive definition)
- Make sure you show how you use the Inductive Hypothesis
- Make sure you justify every step (unless it is only arithmetic)

Recall the Fibonacci numbers

Definition of function $f : \mathbb{Z}^+ \to \mathbb{Z}$:

- $f(1) = f(2) = 1$ and
- $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$

How quickly does $f(n)$ grow?

Growth of Fibonacci numbers

Let's calculate some values

$f(1) = 1$
$f(2) = 1$
$f(3) = f(2) + f(1) = 2$
$f(4) = f(3) + f(2) = 2 + 1 = 3$
$f(5) = f(4) + f(3) = 3 + 2 = 5$
$f(6) = f(5) + f(4) = 8$
$f(7) = f(6) + f(5) = 13$
$f(8) = f(7) + f(6) = 21$
$f(9) = f(8) + f(7) = 34$
$f(10) = f(9) + f(8) = 55$

Question: Is $f(n) \geq 2n$ for all $n \geq 8$?

Proving properties about Fibonacci numbers

Definition of function $f : \mathbb{Z}^+ \to \mathbb{Z}$:

- $f(1) = f(2) = 1$ and
- $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$

We wish to prove $f(n) \geq 2n$ for $n \geq 8$.

We check the first few cases...

$f(8) = f(7) + f(6) = 21$
$f(9) = f(8) + f(7) = 34$
$f(10) = f(9) + f(8) = 55$

So the statement holds for $n = 8, 9, 10$.

Proving $f(n) \geq 2n$ for $n \geq 8$

Definition of function $f : \mathbb{Z}^+ \to \mathbb{Z}$:

- $f(1) = f(2) = 1$ and
- $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$

We wish to prove $f(n) \geq 2n$ for $n \geq 8$.

Let $P(k)$ be the statement that $f(k) \geq 2k$.

Base cases: we already have shown $P(N)$ is true for $N = 8, 9, 10$.
Let $N \geq 10$ be arbitrary (note that I pick 10 because that is the largest base case I examined).
Our Inductive Hypothesis is that $P(k)$ is true for all $k, 8 \leq k \leq N$.

Proving $f(n) \geq 2n$ for $n \geq 8$

Recall that $N \geq 10$ is arbitrary.

The Inductive Hypothesis is

- $P(k) : f(k) \geq 2k$ for all integers k between 8 and N.

The Inductive Step is to show that

$$[P(8) \wedge P(9) \wedge \ldots \wedge P(N)] \rightarrow P(N+1)$$

We write down what $P(N+1)$ asserts to help us come up with the proof!

$$P(N+1) \textit{ asserts that } f(N+1) \geq 2(N+1)$$

In other words, we want to use the I.H. to prove that $f(N+1) \geq 2(N+1)$.

Proving $f(n) \geq 2n$ for $n \geq 8$

The Inductive Hypothesis is

- $P(8) \wedge P(9) \wedge \ldots \wedge P(N)$ is true

where $P(k)$ asserts that $f(k) \geq 2k$ and $N \in \mathbb{Z}^{\geq 10}$ is arbitrary.

Since $N \geq 10$, by the definition of the function f:

- $f(N+1) = f(N) + f(N-1)$

Note that $N-1$ and $N-2$ are both at least 8. Therefore, by the I.H.:

- $f(N) \geq 2N$ and
- $f(N-1) \geq 2(N-1)$

Hence $f(N+1) \geq 2N + 2(N-1) \geq 4N - 2 \geq 2(N+1)$

Note: the second inequality uses that $N \geq 10$.

Hence $f(N+1) \geq 2(N+1)$, which is the same as $P(N+1)$.

Since $N \geq 10$ was arbitrary, $P(N)$ is true for all $N \geq 8$.

Another Strong Induction problem

Let $g : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $g(1) = 1$
- $g(2) = 3$
- $g(n) = g(n-2)$ if $n \geq 3$

For this function g:

- Write down $g(n)$ for all $n = 1, 2, \ldots, 10$
- Guess a closed form solution for $g(n)$ (use your base cases!)
- What is your inductive hypothesis? (State this carefully!)
- Prove your closed form solution true by strong induction.

Another Strong Induction Problem

Let $g : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by $g(1) = 1$, $g(2) = 3$, and $g(n) = g(n-2)$ if $n \geq 3$.

For this function g:

- Write down $g(n)$ for all $n = 1, 2, \ldots, 10$
  - $1, 3, 1, 3, 1, 3, 1, 3, 1, 3$
- Guess a closed form solution for $g(n)$
  - $g(n) = 1$ if n is odd and $g(n) = 3$ if n is even
- What is your inductive hypothesis?
  - Let $P(n)$ denote $g(n) = 1$ if n is odd and $g(n) = 3$ if n is even.
  - Our Inductive Hypothesis is that $P(n)$ is true for all $n, 1 \leq n \leq N$.
  - We will try to infer that $P(N+1)$ is true.

Proving closed form solution for $g(n)$

Note: we came up with the closed form solution by computing $g(n)$ for $n = 1, 2, 3, 4, \ldots, 10$.

Hence, we have verified that $P(n)$ is true for $n = 1, 2, \ldots, 10$.

These are our base cases! (And no, we didn't need them all.)

So we can assume $N \geq 10$.

Proving closed form solution for $g(n)$

Recall $g(1) = 1$, $g(2) = 3$, and $g(n) = g(n-2)$ if $n \geq 3$.

We let $P(n)$ denote the statement: $g(n) = 1$ if n is odd and $g(n) = 3$ if n is even.

We wish to prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Let $N \geq 10$ be arbitrary.

Our Inductive Hypothesis is that $P(1) \wedge P(2) \wedge \ldots \wedge P(N)$ is true.

We wish to infer that $P(N+1)$ is true.

We write down what $P(N+1)$ asserts:

> *$g(N+1) = 1$ if $N+1$ is odd and $g(N+1) = 3$ if $N+1$ is even.*

Proving closed form solution for $g(n)$

Since $N \geq 10$, $N+1 \geq 11$.
Therefore, by the definition of the function g, we have:

$$g(N+1) = g(N+1-2) = g(N-1)$$

Note that $9 \leq N-1 < N$ so that by the Inductive Hypothesis, $P(N-1)$ is true.
Hence $g(N-1) = 1$ if $N-1$ is odd and $g(N-1) = 3$ if $N-1$ is even.

But note that

- $N-1$ and $N+1$ have the same parity (both are odd or both are even)

Therefore, we have established that:

- $g(N+1) = 1$ if $N+1$ is odd and
- $g(N+1) = 3$ if $N+1$ is even

In other words, we have shown that $P(N+1)$ is true.
Since $N \geq 10$ was arbitrary, $P(N)$ is true for all $n \in \mathbb{Z}^+$.

Why would simple induction have failed?

Note that $g(N+1)$ is defined in terms of $g(N-1)$, so we need a statement that is true about $g(N-1)$ and not just about $g(N)$.

In other words, $P(N)$ is not enough to establish $P(N+1)$.

We also needed $P(N-1)$.

Deciding when Strong Induction is necessary

Suppose you are asked to prove a theorem about a recursively defined set or function. If that definition depends only on the previous value, then simple induction will work. Otherwise you probably need strong induction.
Suppose you are asked to establish a bound on each of the following functions; would you need strong induction, or would weak induction suffice?

1. $f(1) = 0, f(n) = 3f(n-1) + 1$ if $n \geq 2$
2. $g(1) = 0, g(2) = 3, g(n) = 2g(n-1) + g(n-2)$ if $n \geq 3$
3. $h(1) = 3, h(2) = 3, h(n) = h(n-1) + h(n-2)$ if $n \geq 3$
4. $k(3) = 4, k(4) = 5, k(n) = k(n-2)$ if $n \geq 5$
5. $p(1) = True, p(2) = False, p(n) = p(n-1) \vee p(n-2)$ if $n \geq 3$
6. $q(1) = True, q(2) = False, q(n) = \neg q(n-1)$ if $n \geq 3$

Class Exercise

Let $f : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $f(1) = 1$
- $f(n) = 1 + \sum_{i=1}^{n-1} f(i)$ if $n \geq 2$

For example, $f(2) = 1 + f(1) = 2$.
Do the following:

- Compute $f(3), f(4)$, and $f(5)$.
- Come up with a closed form solution for $f(n)$.
- Prove it correct by strong induction on n.

Strong Induction

Points:

- Helpful to always state what you want to prove as a boolean statement, $P(n)$, that depends on a parameter n
- Explicitly check the base cases
- Explicitly write down your Inductive Hypothesis: For example, "Our Inductive Hypothesis is that $P(1) \wedge P(2) \ldots \wedge P(N)$ is true for some arbitrary $N \geq n_1$" (where n_1 is the largest base case you checked)
- Make sure your proof uses the information in your problem (e.g., if you are given a recursively defined function, use the its recursive definition)
- Make sure you show how you use the Inductive Hypothesis
- Make sure you justify every step (unless it is only arithmetic)

Next class

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n - 1, m) + f(n, m - 1)$, otherwise

For this function f:

- Compute $f(i, j)$ for all i, j with $1 \leq i, j \leq 3$
- See if you can prove $f(i, j) \geq i + j$ by induction.