CS173
More Complicated Induction Proofs

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Today

We will cover

- Info about Midterm #1
- Info about CS 196
- Review of weak and strong induction
- Proving statements about two variables using induction

Midterm #1

Midterm #1 is October 11, 7:15 PM to 8:30 PM

Please see the course website for how to request a make-up exam time (deadline is October 2, noon)

Midterm 1 will have four parts:

- One proof by weak induction
- One proof by strong induction
- One proof by contradiction
- Multiple choice problems (covering everything)

See http://tandy.cs.illinois.edu/173-2018-midterm1-prep.pdf for some sample problems.

CS 196

First homework due tomorrow via Moodle!

CS 196 website: http://tandy.cs.illinois.edu/CS196-2018.html

If you are registered for CS 196, please email me to let me know if you would like to have regular times to meet as a group with me.

Weak Induction vs. Strong Induction

- Weak Induction asserts a property $P(n)$ for one value of n (however arbitrary)
- Strong Induction asserts a property $P(k)$ is true for all values of k starting with a base case n_0 and up to some final value n.
- The same formulation for $P(n)$ is usually good - the difference is whether you assume it is true for just one value of n or an entire range of values.

Sometimes Strong Induction is needed.

Recurrence relations

Recurrence relations are generally functions defined recursively:

1. $g(1) = 3$ and $g(n) = 3 + g(n-1)$ for $n \geq 2$
2. $f(1) = f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$.

Note that $f(n)$ depends on $f(n-1)$ and $f(n-2)$.

Hence you *must* use strong induction for anything you want to prove about $f(n)$, but you *could* have used weak induction for $g(n)$.

Strong induction is always valid, so practice using it.

Functions of two variables

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

Class exercise: Compute $f(1, 3)$ and $f(2, 2)$

Functions of two variables

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

Why is $f(1, 3) = 4$?

- Because we use the first bullet to compute $f(1, 3)$, and we get $f(1, 3) = 1 + 3 = 4$

Why is $f(2, 2) = 6$?

- Because we use the second bullet to compute $f(2, 2)$, and we get $f(2, 2) = f(1, 2) + f(2, 1)$.
- Also, $f(1, 2) = 1 + 2 = 3$ and $f(2, 1) = 2 + 1 = 3$.
- Therefore $f(2, 2) = 3 + 3 = 6$.

Class Exercise

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n - 1, m) + f(n, m - 1)$, otherwise

For this function f:

- Compute $f(i, j)$ for all i, j with $1 \leq i, j \leq 3$
- See if you can prove $f(i, j) \geq i + j$

Using induction to prove theorems about recursive functions of two variables

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

We would like to prove that $f(n, m) \geq n + m$ for all $n \geq 1$ and $m \geq 1$.

Base cases: $n = 1$ or $m = 1$ follows immediately. So we prove the rest by induction.

What is our inductive hypothesis?

Inductive hypothesis

Recall that

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

What happens if we try to do induction on n?

Inductive hypothesis, continued

Recall that

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

We can't do induction on n because $f(n, m)$ depends on $f(n, m-1)$.

We also can't do induction on m because $f(n, m)$ depends on $f(n-1, m)$.

Inductive hypothesis, continued

Recall that

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n - 1, m) + f(n, m - 1)$, otherwise

We need a value that goes down... so that $f(n, m)$ depends on values *to which the inductive hypothesis can be applied.*

What value goes down?

Inductive hypothesis, continued

Recall that

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

The sum of the parameters goes down!

Inductive hypothesis, continued

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

So, our inductive hyopthesis will be:

$P(K) : f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$

Note that we are inducing on K, and defining K to be the sum of the parameters to the function f.

The base case

Recall $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, and the inductive hyopthesis is:

$P(K) : f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$

The smallest value for $n + m$ is 2; hence, the base case is $K = 2$.

When $K = 2$, $n = m = 1$ and the statement holds.

Finishing the Induction Proof

Recall $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, and the inductive hyopthesis is:

> $P(K) : f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$

To finish the induction proof, we need to show $P(K) \to P(K+1)$, which is equivalent to showing

$$P(K) \to \forall n, m \text{ such that } n + m \leq K + 1, f(n, m) \geq n + m$$

However, since the I.H. assumes $f(n, m) \geq n + m$ whenever $n + m \leq K$, we only need to show that

- $P(K) \to f(n, m) \geq n + m$ whenever $n + m = K + 1$.

Another induction proof, continued

Let n, m be given so that $n + m = K + 1$.

If $n = 1$ or $m = 1$, then by definition $f(n, m) = n + m$, and the statement holds.
So assume $n \geq 2$ and $m \geq 2$, so that

$$f(n, m) = f(n-1, m) + f(n, m-1)$$

Note that $n + m = K + 1$ and so $n + m - 1 = K$.

Hence we can apply the Inductive Hypothesis to $f(n-1, m)$ and $f(n, m-1)$.

Therefore,

$$f(n-1, m) \geq n + m - 1$$

and

$$f(n, m-1) \geq n + m - 1$$

Hence

$$f(n, m) = f(n-1, m) + f(n, m-1) \geq 2(n + m - 1)$$

Another induction proof, continued

So far we have shown that when n, m are both at least 2 and $n + m = K + 1$, then

$$f(n, m) = f(n-1, m) + f(n, m-1) \geq 2(n + m - 1)$$

However, a little more arithmetic finishes this!

$$f(n, m) \geq 2(n + m - 1) = n + m + (n + m - 2) \geq n + m$$

since $n + m - 2 \geq 0$.

Since K was arbitrary, the statement holds for all $K \geq 2$, and hence for all pairs of positive integers n, m that sum to K.

This is what we wanted to prove. Q.E.D.

Summarizing what we did

Recall that we had a recursively defined function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n - 1, m) + f(n, m - 1)$, otherwise

We wanted to prove that $f(m, n) \geq m + n$ for all positive integers m, n.

It is easy to verify this inequality for the case where $m = 1$ or $n = 1$. To prove it true for all m, n, we used induction.

But induction must be done for some single parameter.

We used $K = m + n$ as our single parameter.

Our inductive hyopthesis was

> $P(K) : f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$

Class exercise

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $f(1, x) = x = f(x, 1)$ for all $x \in \mathbb{Z}^+$
- $f(a, b) = \max\{f(a-1, b) + b, f(a, b-1) + a\}$ if $a \geq 2$ and $b \geq 2$

Compute $f(a, b)$ for all a, b with $1 \leq a, b \leq 3$.

What do you think the closed form solution should be?

What will your Inductive Hypothesis be?

At home: use induction to prove your closed form solution correct. (Note: do you need stong induction?)

What we learned

We learned:

- The base case is sometimes more than one value.
- We learned about the difference between strong and weak induction, and that strong induction is always at least as powerful as weak induction.
- The inductive hypothesis is sometimes not on an obvious parameter, but on something defined using obvious parameters (like the sum).
- Induction can be used to prove properties about recursively defined functions and sets.