

Eulerian Graphs

<https://www.math.ku.edu/~jmartin/courses/math105-F11/Lectures/chapter5-part2.pdf>

by Jeremy Martin, Professor of Mathematics,
University of Kansas.

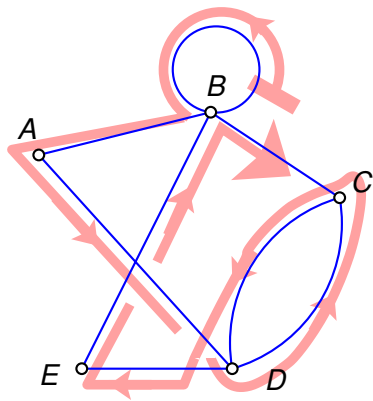
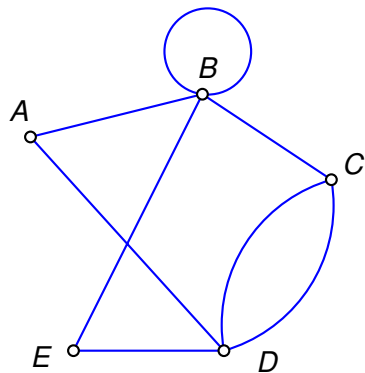
Euler Paths and Euler Circuits

An **Euler path** is a path that uses every edge of a graph exactly once.

An **Euler circuit** is a circuit that uses every edge of a graph exactly once.

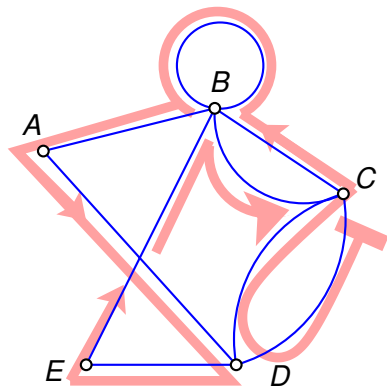
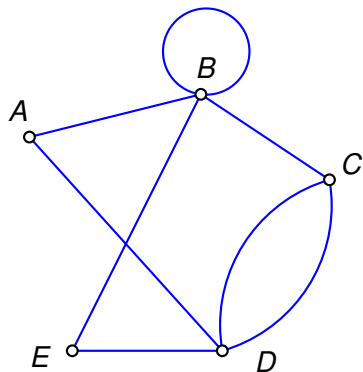
- ▶ An Euler path starts and ends at **different** vertices.
- ▶ An Euler circuit starts and ends at **the same** vertex.

Euler Paths and Euler Circuits



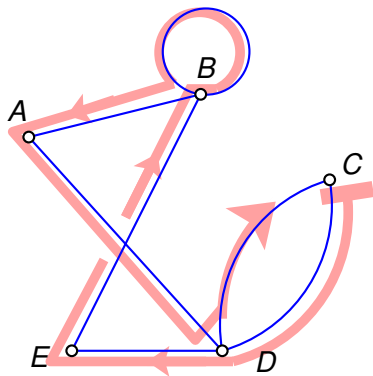
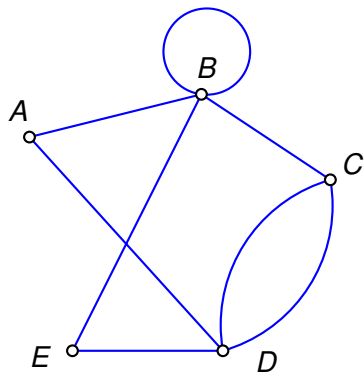
An Euler path: **BBADCDEBC**

Euler Paths and Euler Circuits



An Euler circuit: CDCBBADEBC

Euler Paths and Euler Circuits



Another Euler circuit: **CDEBBADC**

Euler Paths and Euler Circuits

Is it possible to determine whether a graph has an Euler path or an Euler circuit, without necessarily having to find one explicitly?



The Criterion for Euler Paths

Suppose that a graph has an Euler path P .

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Therefore, there are $2s$ edges having v as an endpoint.

The Criterion for Euler Paths

Suppose that a graph has an Euler path P .

For every vertex v other than the starting and ending vertices, the path P enters v the same number of times that it leaves v (say s times).

Therefore, there are $2s$ edges having v as an endpoint.

Therefore, all vertices other than the two endpoints of P must be even vertices.

The Criterion for Euler Paths

Suppose the Euler path P starts at vertex x and ends at y .

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Then P leaves x one more time than it enters, and leaves y one fewer time than it enters.

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Then P leaves x one more time than it enters, and leaves y one fewer time than it enters.

Therefore, **the two endpoints of P must be odd vertices.**

The Criterion for Euler Paths

The inescapable conclusion (“based on reason alone!”):

If a graph G has an Euler path, then it must have exactly two odd vertices.

Or, to put it another way,

If the number of odd vertices in G is anything other than 2, then G cannot have an Euler path.

The Criterion for Euler Circuits

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- ▶ The circuit C **enters** v the same number of times that it **leaves** v (say s times), so v has degree $2s$.
- ▶ That is, **v must be an even vertex.**

The Criterion for Euler Circuits


The inescapable conclusion (“based on reason alone”):

If a graph G has an Euler circuit, then all of its vertices must be even vertices.

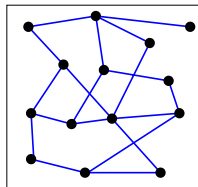
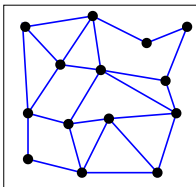
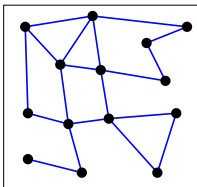
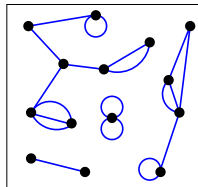
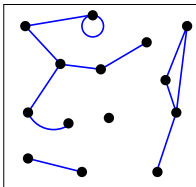
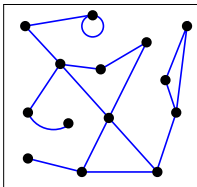
Or, to put it another way,

If the number of odd vertices in G is anything other than 0, then G cannot have an Euler circuit.

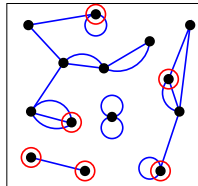
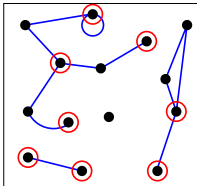
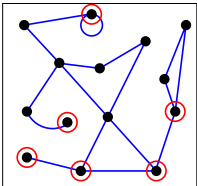
Things You Should Be Wondering

- ▶ Does **every** graph with **zero** odd vertices have an Euler circuit?
- ▶ Does **every** graph with **two** odd vertices have an Euler path?
- ▶ Is it possible for a graph have just **one** odd vertex? 

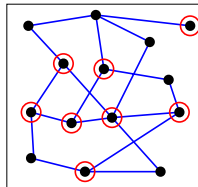
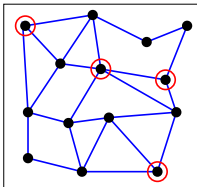
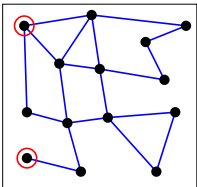
How Many Odd Vertices?



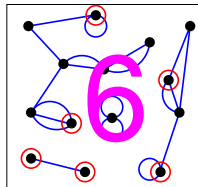
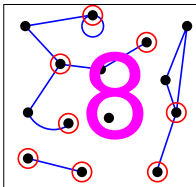
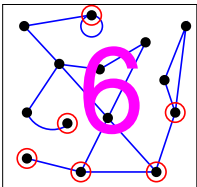
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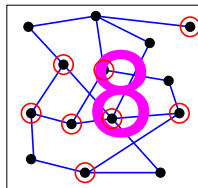
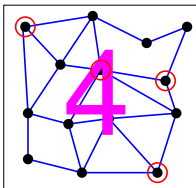
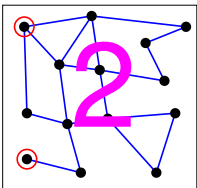
Odd vertices



How Many Odd Vertices?



Number of odd vertices



The Handshaking Theorem

The Handshaking Theorem says that

In every graph, the sum of the degrees of all vertices equals twice the number of edges.

If there are n vertices V_1, \dots, V_n , with degrees d_1, \dots, d_n , and there are e edges, then

$$d_1 + d_2 + \cdots + d_{n-1} + d_n = 2e$$

Or, equivalently,

$$e = \frac{d_1 + d_2 + \cdots + d_{n-1} + d_n}{2}$$

The Handshaking Theorem

Why “Handshaking”?

If n people shake hands, and the i^{th} person shakes hands d_i times, then the total number of handshakes that take place is

$$\frac{d_1 + d_2 + \cdots + d_{n-1} + d_n}{2}.$$

(How come? Each handshake involves two people, so the number $d_1 + d_2 + \cdots + d_{n-1} + d_n$ counts every handshake twice.)

The Number of Odd Vertices

- ▶ The number of edges in a graph is

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- ▶ Therefore, $d_1 + d_2 + \cdots + d_n$ must be an **even number**.
- ▶ Therefore, the numbers d_1, d_2, \cdots, d_n must include an **even number of odd numbers**.
- ▶ **Every graph has an even number of odd vertices!**

Back to Euler Paths and Circuits

Here's what we know so far:

# odd vertices	Euler path?	Euler circuit?
0	No	Maybe
2	Maybe	No
4, 6, 8, ...	No	No
<i>1, 3, 5, ...</i>	<i>No such graphs exist!</i>	

Can we give a better answer than “maybe”?

Euler Paths and Circuits — The Last Word

Here is the answer Euler gave:

# odd vertices	Euler path?	Euler circuit?
0	No	Yes*
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* *Provided the graph is connected.*

Euler Paths and Circuits — The Last Word


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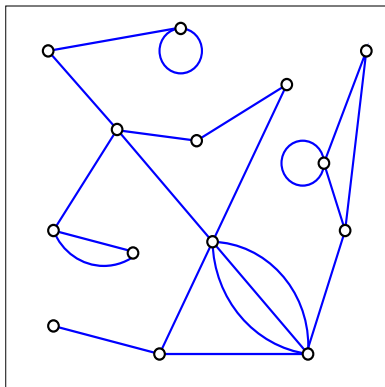
Next question: If an Euler path or circuit exists, how do you find it?

Bridges

Removing a single edge from a connected graph can make it disconnected. Such an edge is called a **bridge**. 

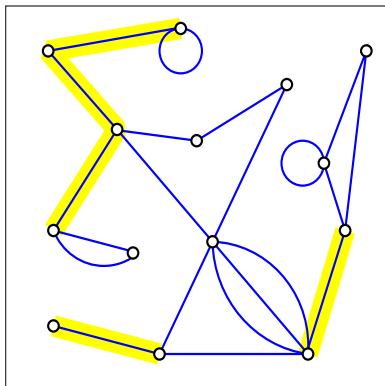
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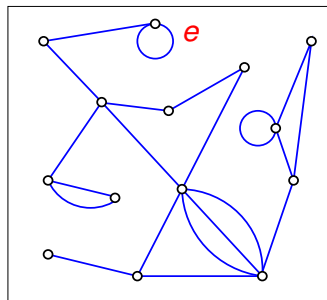
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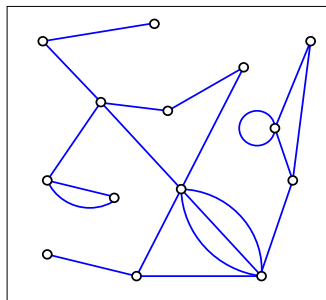


Bridges

Loops cannot be bridges, because removing a loop from a graph cannot make it disconnected.

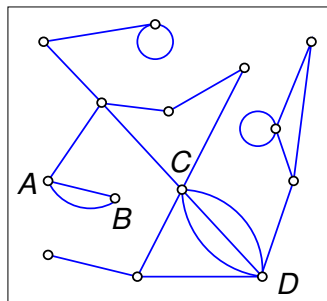


*delete
loop e*

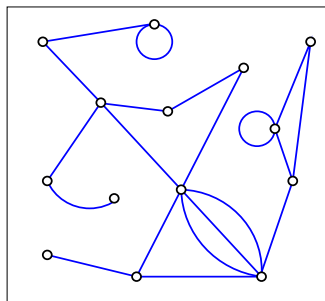


Bridges

If two or more edges share both endpoints, then removing any one of them cannot make the graph disconnected. Therefore, none of those edges is a bridge.

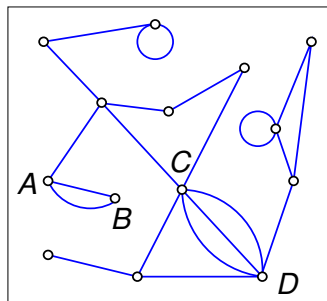


*delete
multiple
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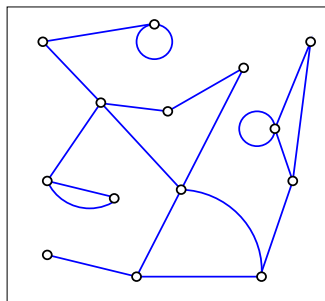


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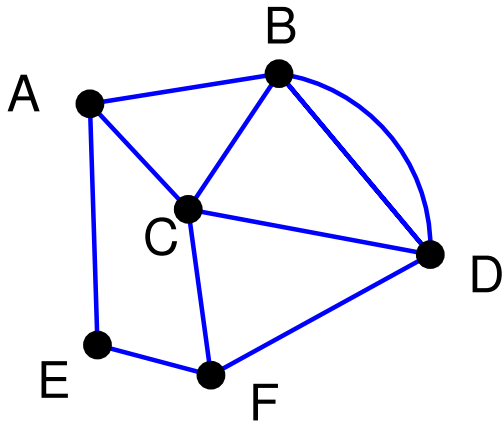


Finding Euler Circuits and Paths

“Don't burn your bridges.”

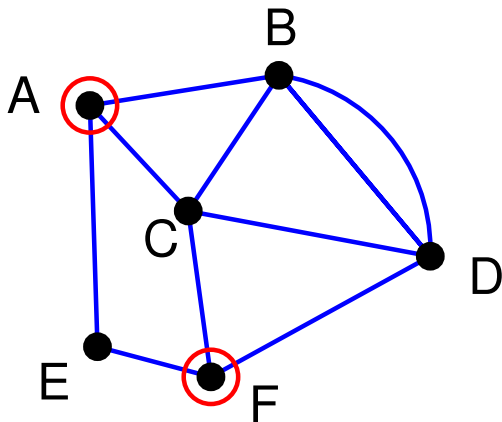
Finding Euler Circuits and Paths

Problem: Find an Euler circuit in the graph below.



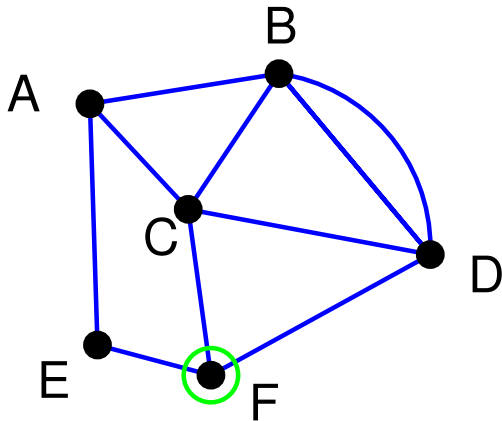
Finding Euler Circuits and Paths

There are two odd vertices, A and F. Let's start at F.



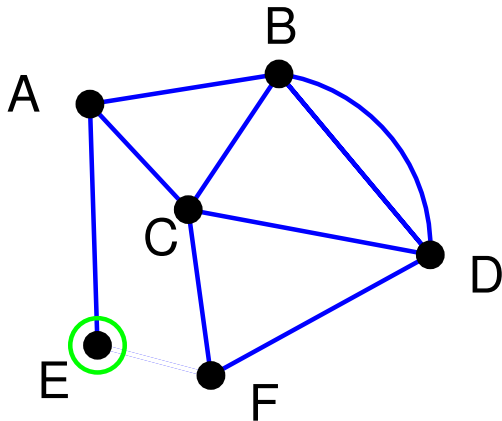
Finding Euler Circuits and Paths

Start walking at F. When you use an edge, delete it.



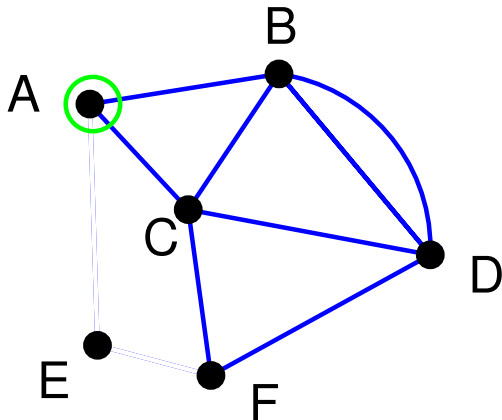
Finding Euler Circuits and Paths

Path so far: FE



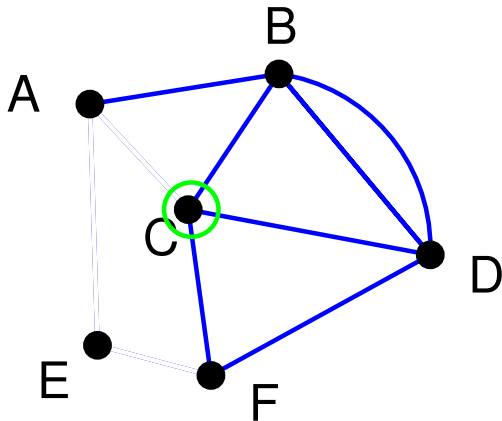
Finding Euler Circuits and Paths

Path so far: FEA



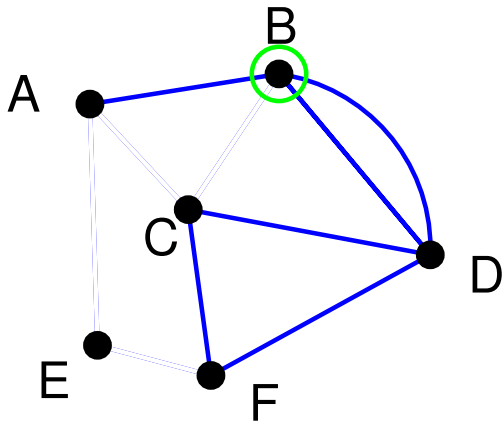
Finding Euler Circuits and Paths

Path so far: FEAC



Finding Euler Circuits and Paths

Path so far: FEACB



Finding Euler Circuits and Paths

Up until this point, the choices didn't matter.

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But now, crossing the edge BA would be a mistake, because we would be stuck there.

Finding Euler Circuits and Paths

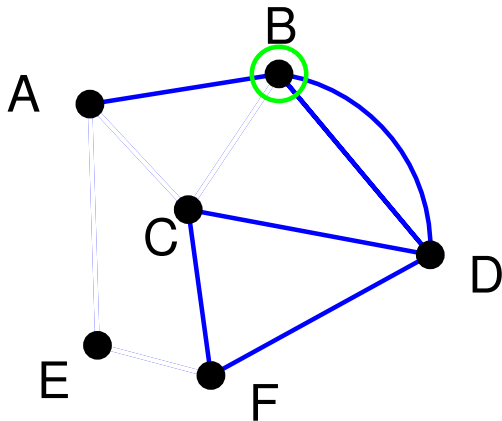
Up until this point, the choices didn't matter.

But now, crossing the edge BA would be a mistake, because we would be stuck there.

The reason is that BA is a **bridge**. We don't want to cross (“burn”?) a bridge unless it is the only edge available.

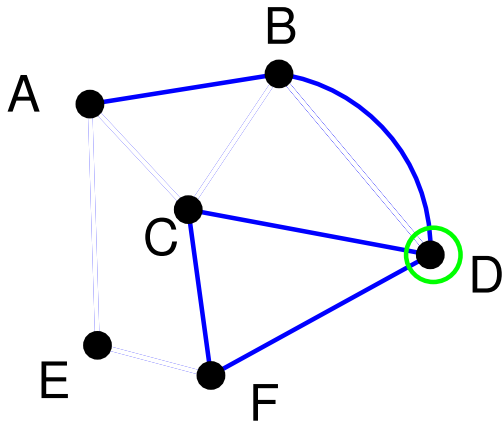
Finding Euler Circuits and Paths

Path so far: FEACB



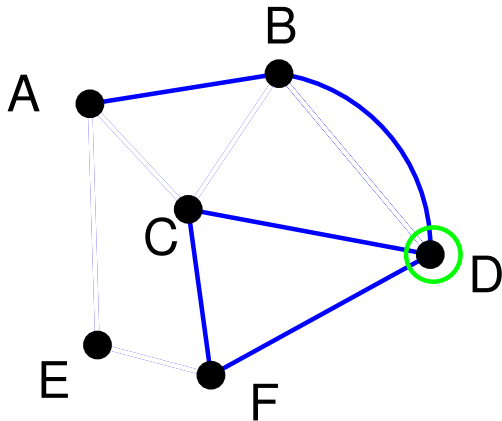
Finding Euler Circuits and Paths

Path so far: FEACBD.



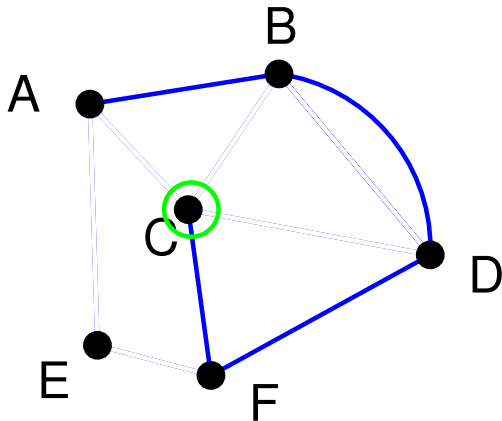
Finding Euler Circuits and Paths

Path so far: FEACBD. **Don't cross the bridge!**



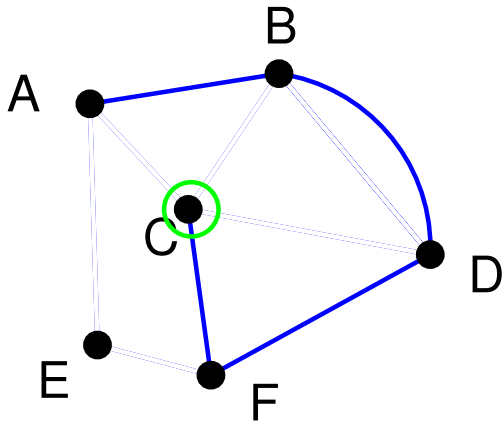
Finding Euler Circuits and Paths

Path so far: FEACBDC



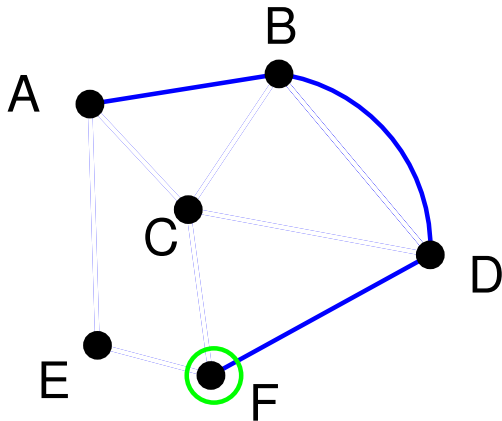
Finding Euler Circuits and Paths

Path so far: FEACBDC Now we have to cross the bridge CF.



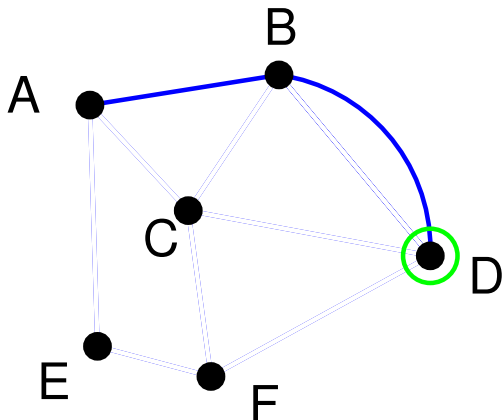
Finding Euler Circuits and Paths

Path so far: FEACBDCF



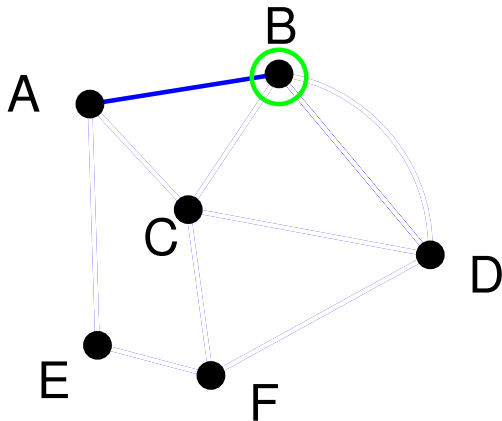
Finding Euler Circuits and Paths

Path so far: FEACBDCFD



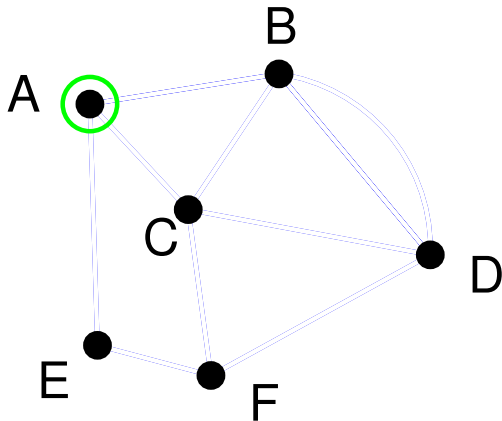
Finding Euler Circuits and Paths

Path so far: FEACBDCFDB



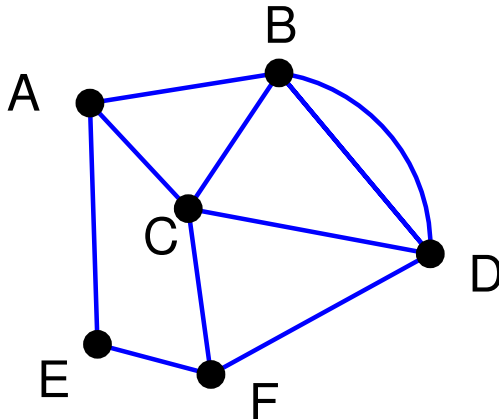
Finding Euler Circuits and Paths

Euler Path: FEACBDCFDDBA



Finding Euler Circuits and Paths

Euler Path: FEACBDCFDDBA



Fleury's Algorithm

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This is called **Fleury's algorithm**, and it always works!

Fleury's Algorithm: Another Example

