CS466

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December 26, 2016

CS 466, Spring 2017
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Dynamic Programming

Dynamic programming is an algorithmic design technique that can make it easy to solve problems efficiently.
Dynamic programming is similar to recursion – but it is bottom-up, instead of top-down.

Fibonacci numbers

Consider how to calculate Fibonacci numbers, $F(n)$, defined by

- $F(1) = F(2) = 1$
- $F(n) = F(n-1) + F(n-2)$ if $n > 2$

Let's do this calculation recursively.
Input: $n \in Z^+$
Algorithm:

- If $[n = 1$ or $n = 2]$ then Return(1)
- Else
  - Recursively compute $F(n-1)$ and store in X
  - Recursively compute $F(n-2)$ and store in Y
  - Return$(X + Y)$

Running time of recursive algorithm for F(n)

The running time $t_1(n)$ of this algorithm satisfies:

- $t_1(1) = C$
- $t_1(2) = C$
- $t_1(n) = t_1(n-1) + t_1(n-2) + C'$

for some positive integers C, C'.

It's immediately obvious that $t_1(n) \geq F(n)$ for all $n \in Z^+$ (compare the recurrence relations).

This is a problem, because $F(n)$ grows exponentially (look at http://mathworld.wolfram.com/FibonacciNumber.html), and so $t_1(n)$ grows at least exponentially!

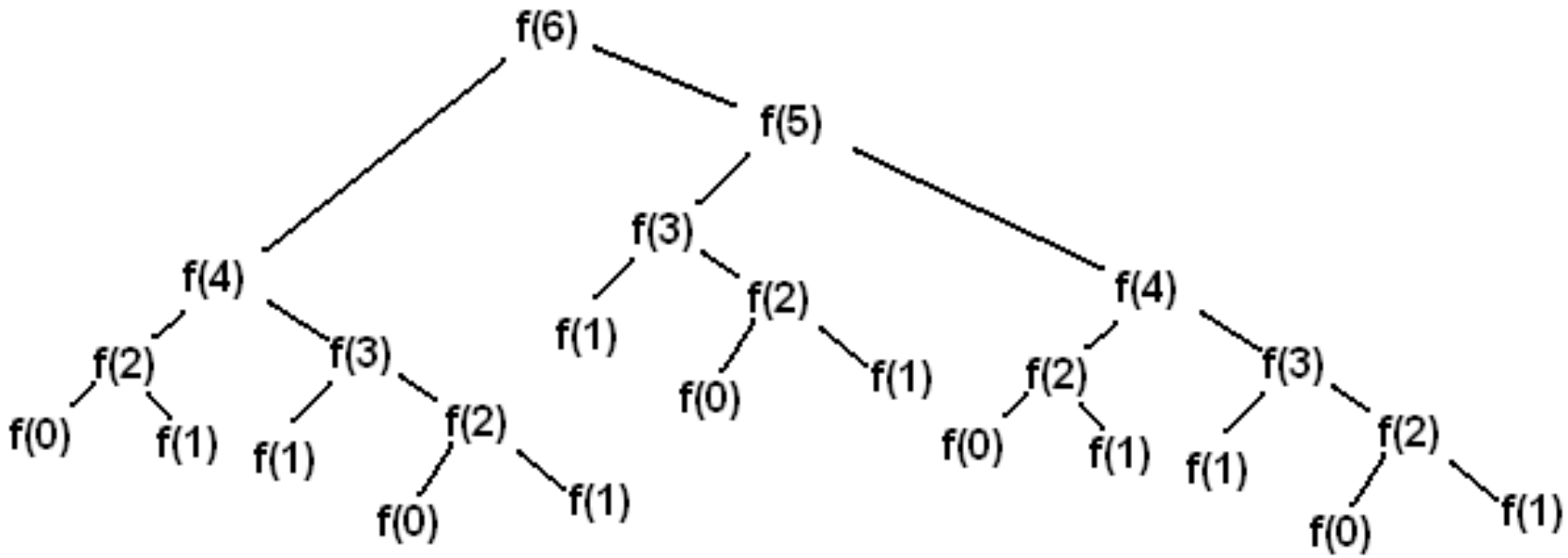
Recursive computation of the Fibonacci numbers

When we compute the Fibonacci numbers recursively, we compute $F(n)$ by independently computing $F(n-1)$ and $F(n-2)$.

Note that $F(n-1)$ also requires that we compute $F(n-2)$, and so $F(n-2)$ is computed twice, rather than once and then re-used.

It would be much better if we had stored the computations for $F(i)$ for smaller values of i (in $\{1, 2, \ldots, n-1\}$) so that they could be re-used.

The recursion tree for the Fibonacci numbers



https://en.wikibooks.org/wiki/Algorithms/Dynamic_Programming

A better way of computing $F(n)$

The simple recursive way of computing $F(n)$ is exponential, but there is a very simple **dynamic programming** approach that runs in linear time!

Input: $n \in Z^+$

- If $n \leq 2$ return 1. Else:
  - $F[1] := 1$
  - $F[2] := 1$
  - For $i = 3$ upto n, DO
    - $F[i] := F[i-1] + F[i-2]$

    Endfor
  - Return($F[n]$)

The running time $t_2(n)$ for this algorithm satisfies

- $t_2(1) \leq C'$
- $t_2(2) \leq C'$
- $t_2(n) = t_2(n-1) + C'$ if $n > 2$

for some positive constant C'. Note the difference in the recursive definition for $t_1(n)$ and $t_2(n)$.

Bounding this recurrence relation

The running time $t_2(n)$ for this algorithm satisfies

- $t_2(1) \leq C_0$
- $t_2(2) \leq C_1$
- $t_2(n) \leq t_2(n-1) + C_2$

for some positive constants C_0, C_1, and C_2.
Let $C' = \max\{C_0, C_1, C_2\}$. It is easy to see that $C' > 0$.

We will prove that $t_2(n) \leq C'n$ for all $n \geq 1$, by induction on n.

Running time analysis of the DP algorithm for $F(n)$

Note that

- $t_2(n) \leq C'$ if $n \leq 2$ and
- $t_2(n) \leq t_2(n-1) + C'$ for $n > 2$

We prove by induction that $t_2(n) \leq C'n$ for all $n \in Z^+$.
Proof: The base case is $n = 1, 2$, and is verified.
The inductive hypothesis is that there is some $K \in Z^+$ with $K \geq 2$, such that for all $n \in \{1, 2, \ldots, K\}$, $t_2(n) \leq C'n$.
We want to show that $t_2(K+1) \leq C'(K+1)$.
Since $K \geq 2$, $K + 1 > 2$. Hence, $t_2(K+1) = t_2(K) + C'$.
By the inductive hypothesis, $t_2(K) \leq C'K$. Hence,

$$t_2(K+1) \leq t_2(K) + C'$$

$$\leq C'K + C' = C'(K+1)$$

which is what we wanted to prove.
Since K was arbitrary, it follows that $t_2(n) \leq C'n$ for all $n \in Z^+$.

Finding a longest increasing subsequence

Input: string $X = x_1 x_2 \ldots x_n$ over alphabet Σ
Output: increasing subsequence of X that is as long as possible.
Note, we require that the subsequence be strictly increasing!

Example: $X = 1, 3, 1, 8, 2, 4, 9, 2, 10, 3$
What are some increasing subsequences?
$1, 3, 8, 9, 10$ is an increasing subsequence. Is it the longest one?
Is $1, 1$ an increasing subsequence? (Answer: NO!)

Dynamic programming algorithm for LIS

Let $n[i]$ be the length of the longest increasing subsequence of the first i letters of X, and that includes x_i.
For example, when $X = 1, 3, 1, 8, 2, 4, 9, 2, 10, 3$, then

- $n[1] = 1$
- $n[2] = 2$
- $n[3] = 1$
- $n[4] = 3$
- $n[5] = 2$

After we compute $n[i]$ for $i = 1, 2, \ldots, n$, then we set
$LIS = \max\{n[i] : i = 1, 2, \ldots, n\}$.
How can we compute $n[i]$ efficiently?

Dynamic programming algorithm for LIS

To compute $n[i], i = 1, 2, \ldots, n$, we could use recursion or dynamic programming. Let's do it using dynamic programming.

- For $i = 1$ upto n, DO
  - Compute $n[i]$ somehow

  Endfor

Can we use the values for $n[1], n[2], \ldots, n[i-1]$ to compute $n[i]$?

Computing $n[i]$

Can we use the values for $n[1], n[2], \ldots, n[i-1]$ to compute $n[i]$?
Let's define $Smaller[i] = \{j \in \{1, 2, \ldots, i-1\} : x_i > x_j\}$.

- Case: $Smaller[i] = \emptyset$. Then $n[i] = 1$, because the only increasing subsequence of $x_1 x_2 \ldots x_i$ that includes x_i is x_i itself.
- Case: $Smaller[i] \neq \emptyset$. Then, for every $j \in Smaller[i]$, there is an increasing subsequence of $x_1 x_2 \ldots x_i$ of length $n[j] + 1$. Hence, $n[i] = \max\{n[j] + 1 : j \in Smaller[i]\}$.

DP algorithm

- For $i = 1$ upto n, DO
  - Comment: Compute $Smaller[i]$
    If $Smaller[i] = \emptyset$, then $n[i] := 1$
    else $n[i] := \max\{n[j] + 1 : j \in Smaller[i]\}$

  Endfor

Running time analysis:

- Computing each $Smaller[i] : i = 1, 2, \ldots, n$ takes $O(n)$ time, and so computing them all takes $O(n^2)$ time.
- Computing each $n[i]$ after all the previous $n[j]$ are computed takes $O(i)$ time. Since $i \leq n$, this is $O(n)$ for each i. Hence, these calculations take $O(n^2)$ overall.

Altogether, the running time is $O(n^2)$ time.

Two-person games

Remember the original two person game?

- There are n rocks on pile 1, and m rocks on pile 2.
- Each player can take one rock off of one pile, or one rock off of each pile.
- The person to take the last rock off wins.

Let's write a dynamic programming algorithm to determine who has a winning strategy.

DP vs. Recursive Algorithms

In these examples, the DP approach has been more efficient than recursion. But this is not always the case!

Sometimes the recursive approach is faster.

It depends on whether you really need to compute *all* the subproblems. If you do, then DP is at least as efficient, and often faster.

Writing DP algorithms

Please observe the following guidelines for writing a dynamic programming algorithm:

- Explain your variables using English, showing what they are supposed to mean
- Show how to compute the values for the boundary conditions
- Specify the order in which you compute the values
- Show how to compute each value based on the earlier computations
- Show where the final answer is stored

Summary

- Dynamic programming and recursive algorithms are two ways of dealing with algorithm design.
- One is top down (recursion) and the other is bottom-up (dynamic programming).
- You can **prove** your algorithm is correct using induction, when the algorithm uses recursion or dynamic programming.

In both cases, you identify subproblems and show how solving subproblems lets you solve big problems.