CS466, Spring 2017

Tandy Warnow

January 27, 2017

CS 466
Tandy Warnow

Today's material

- Big-O definition (review)
- Proving Big-O by induction for recursively defined functions
- Common pitfalls

Big-O definition

Let f and g be functions from Z^+ (positive integers) to R^+ (positive real numbers).

We say that f is $O(g)$ if there exists constants C and N such that $f(n) \leq Cg(n)$ whenever $n > N$.

To *prove* that f is $O(g)$ it is best to produce the constants C and N.

Bounding recursive functions

We are often given functions f that are defined recursively, and want to find a function g such that f is $O(g)$, and then prove this statement.

- $f(1) = f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$ if $n > 2$.
- $f(1) = 3$ and $f(n) = 2f(n-1)$ if $n > 1$.
- $f(1) = 5$ and $f(n) = 2f(n-1) + 2^n$ if $n > 1$.
- $f(1) = 3$ and $f(n) = 2n + f(n-1)$ if $n > 1$.

An example

Example:

- $f(1) = 3$
- $f(n) = 2n + f(n-1)$ if $n \geq 2$.

We will guess that $f(n)$ is $O(n)$, and try to prove this by induction.

Base case holds (i.e., 3 is $O(n)$).

Inductive Hypothesis: for some arbitrary $n \geq 1$, $f(n)$ is $O(n)$.

Then $f(n+1) = 2(n+1) + O(n)$ which is $O(n)$.

Since n was arbitrary, our assertion is proved.

If we graph $f(n)$ we see it doesn't look like it's bounded from above by a linear function.

- $f(1) = 3$
- $f(2) = 3 + 4 = 7$
- $f(3) = 7 + 6 = 13$

So, this doesn't work, but why not?

Finding a closed form for $f(n)$

Let's find a closed form solution for $f(n)$, defined by $f(1) = 3$ and $f(n) = 2n + f(n-1)$ if $n \geq 2$.

We will guess that $f(n)$ is quadratic, and so is equal to $f(n) = an^2 + bn + c$ for some constants a, b, c.

To solve for a, b, c we need three values:

- $f(1) = 3$
- $f(2) = 7$
- $f(3) == 13$

We solve for a, b, c and get $a = b = c = 1$. In other words, we

guess that

$$f(n) = n^2 + n + 1$$

Proving $f(n)$ is $O(n^2)$ by induction

We will prove $f(n)$ is $O(n^2)$ by induction, but we do this by proving that $f(n) = n^2 + n + 1$.

Base case is $n = 1$, and we obtain $f(1) = 1 + 1 + 1 = 3$, as required.

Inductive Hypothesis: for some arbitrary $N \geq 1$, $f(x) = x^2 + x + 1$

for all $1 \leq x \leq N$.

We would like to prove that $f(N+1) = (N+1)^2 + (N+1) + 1$.

$$f(N+1) = 2(N+1) + f(N) \text{ (by definition)}$$
$$= 2N + 2 + (N^2 + N + 1) \text{ (by the I.H.)}$$
$$= N^2 + 3N + 3 \text{ (by arithmetic)}$$
$$= (N+1)^2 + (N+1) + 1 \text{ (by arithmetic)}$$

Since N was arbitrary, we have established that $f(n) = n^2 + n + 1$ for all $n = 1, 2, 3, \ldots$.

A Common Pitfall

Note:

- We found a closed form solution for $f(n) = n^2 + n + 1$.
- Therefore $f(n)$ is not $O(n)$!
- What was wrong with our first "proof"?

Establishing that f is $O(g)$ depends on a particular pair of constants C and N.

The constants cannot depend on the value of n.

To be safe, always provide the constants C and N, and show they are valid.

Problem 11 from Homework 1

- $f(1) = f(2) = 1$
- $f(n) = f(n-1) + f(n-2)$ if $n > 1$.

We will prove f is $O(g)$ where $g(n) = 2^n$.

Proof: We need to find constants C and N such that $f(n) \leq C2^n$ for all $n > N$.

We examine the first few values of $f(n)$ and get

$$f(1) = f(2) = 1, f(3) = 2, f(4) = 3, f(5) = 5$$

.

Let's try $C = 1, N = 0$. Hence, we want to prove $f(n) \leq 2^n$ for all

$n = 1, 2, \ldots.$

Continuing the proof

The base cases are $n = 1, 2$, which hold easily.

(Weak) Inductive Hypothesis: for some arbitrary $N \geq 2$,
$f(N) \leq 2^N$.
Note:

- We should do this using strong induction, but let's just do weak induction for now.
- We want to derive that $f(N+1) \leq 2^{N+1}$.

Then $f(N+1) = f(N) + f(N-1)$ (by definition, since $N+1 \geq 3$)

$\leq 2^N + 2^{N-1}$ (by the I.H.)
$\leq 2 * 2^N = 2^{N+1}$ (by arithmetic)

Since N was arbitrary, we have proven our assertion. Q.E.D. *What was wrong with this argument?*

We needed Strong Induction!

In our "proof", we said
$f(N+1) = f(N) + f(N-1)$ (by definition)
$\leq 2^N + 2^{N-1}$ (by the I.H.)

We derived this by saying $f(N) \leq 2^N$ and $f(N-1) \leq 2^{N-1}$. But our I.H. was only that $F(N) \leq 2^N$. In other words, we did not also assume that $F(N-1) \leq 2^{N-1}$. So this proof does not work!

A better proof

Let's make a Strong Inductive Hypothesis:

For some $N \geq 2$ and for all $x \in \{1, 2, \ldots, N\}$, $F(x) \leq 2^x$.

If we do this, the assertion that $F(N-1) \leq 2^{N-1}$ follows from the Inductive Hypothesis is fine.

Whenever your recurrence relation depends on more than just the immediate preceeding value, you *really do need strong induction*!

Formulating a Weak Inductive Hypothesis

- Never make your inductive hypothesis what you want to prove. Thus, don't write: "Inductive Hypothesis: for all n, $f(n) \leq 2^n$."
- Be careful not to leave it ambiguous, either. Don't write: "Inductive Hypothesis: $f(n) \leq 2^n$", since the reader might think you mean it's true for all n.
- Better to say "Inductive Hypothesis: for some arbitrary n, $f(n) \leq 2^n$"

Formulating a Strong Inductive Hypothesis

Which of the following make sense?

- For all N and x, $1 \leq x \leq N$, $f(x) \leq 2^x$.
- For some N and x, $1 \leq x \leq N$, $f(x) \leq 2^N$.
- For some $N \geq 2$ and for all $x, 1 \leq x \leq N$, $f(x) \leq 2^x$
- For some N and for all x with $1 \leq x \leq N$, $f(x) \leq 2^x$
- For some N and for all x with $1 \leq x \leq N$, $f(x) \leq 2^N$
- For some N and for all x with $1 \leq x \leq N$, $f(x) \leq 2^x$