CS173
Strong Induction and Functions

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CS 173

Introduction to Strong Induction (also Functions)

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Preview of the class today

- What are functions?
- Weak induction
- Strong induction
- A graph theory question

Graph Theory Question

Consider the following question

> All finite simple graphs with minimum degree at least 1 are connected

Is it true? If so, prove by induction on the number of vertices.

What are functions?

Rosen, Chapter 2.3 (pp. 138-155).

Functions are mappings from one set to another!

The notation $f : X \rightarrow Y$ means:

- f is a function from X to Y
- You can think of f as an algorithm with input coming from X and output in Y
- The output of f on input x is denoted $f(x)$
- In other words $f(x)$ has a value in Y for each element $x \in X$
- Also, $f(x)$ never has two values in Y for any $x \in X$
- X is referred to as the "domain" of f and Y is referred to as the "co-domain" of f.
- The subset of Y defined to be $\{y \in Y | \exists x \in X \ s.t. \ f(x) = y\}$ is called the "range" of f.

Functions, continued

Functions may or may not have nice closed forms.
Consider functions mapping $\mathbb{R} \to \mathbb{R}$:

1. $f(n) = n^2$
2. $f(n) = 1$ if $n \geq 0$ and $f(n) = -1$ if $n < 0$
3. $f(n) = n$ if $n \in \mathbb{Z}$ and $f(n) = \pi$ if $n \in \mathbb{R} \setminus \mathbb{Z}$

Or something like this: $f : \{0, 1, 2, 3\} \to \{3, 5\}$ given by:

- $f(0) = 3$
- $f(1) = 5$
- $f(2) = 3$
- $f(3) = 3$

Question: How many different functions are there from $\{0, 1, 2, 3\}$ to $\{3, 5\}$?

Representing Functions

You can represent a function $F : X \to Y$ with a set of ordered pairs:

- $\{(x, y) | y = f(x), x \in X, y \in Y\}$

Similarly, you can draw this as a directed bipartite graph (X, Y) with directed edges $x \to y$ for each $x \in X, y \in Y$ such that $y = f(x)$.

Are these functions?

Consider each of the following mappings... and determine which ones are functions.

Let People denote the set of all people who have ever lived. For each mapping, consider whether it satisfies the requirements for being a function.

- F is a mapping from People to People that has the ordered pair (x, y) if and only if x is the mother of y
- F is a mapping from People to People that has the ordered pair (x, y) if and only if x is the child of y
- F is a mapping from $\mathbb{R}^+$ to $\mathbb{R}$ that has the ordered pair (x, y) if and only if $y = x^2$.
- F is a mapping from $\mathbb{R}^+$ to $\mathbb{R}$ that has the ordered pair (x, y) if and only if $x = y^2$.

Weak induction

What we did was "weak induction":

- Write down what you want to prove as a statement $P(n)$ that depends on a parameter n
- Show that $P(n)$ is true for some base case(s)
- Show that $P(n) \rightarrow P(n+1)$

Weak induction

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$
- $F(n) = 2F(n-1)$ if $n > 1$

Then $F(n) = 2^n$ for all $n \in \mathbb{Z}^+$.

The proof by weak induction is straightforward.

A harder case

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

Then for all positive integers n, $F(n) = n \mod 2$

Let's try to prove this by induction on n.

Let's make our Boolean statement $P(n)$ be:

- $F(n) = n \mod 2$

where $n \mod 2$ is the remainder after dividing n by 2.

Why simple induction can fail

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

Suppose $P(n)$ is the statement $F(n) = n \mod 2$.

Base cases are fine ($n = 1, n = 2$).

Does $P(n) \to P(n+1)$?

Let $n \geq 2$ and assume $P(n)$ is true.

Why weak induction can fail

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

Suppose $P(n)$ is the statement $F(n) = n \mod 2$.

To show that $P(n) \to P(n+1)$ we need to show that

- If $F(n) = n \mod 2$ then $F(n+1) = n+1 \mod 2$.

Let's try this.

Why weak induction can fail

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

We assume $n \geq 2$ is arbitrary and $P(n)$ is true.

Then $n + 1 \geq 3$ and so (by definition) $F(n+1) = F(n-1)$.

What's our next step?

Why weak induction can fail

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

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Then $n+1 \geq 3$ and so (by definition) $F(n+1) = F(n-1)$.

What's our next step?

Recall the Inductive Hypothesis is $P(n)$: $F(n) = n \mod 2$.

The Inductive Hypothesis tells us nothing about $F(n-1)$.

What do we do now???

Our induction proof has failed!

Our Inductive Hypothesis only told us about $F(n)$, not about $F(n-1)$.

Our induction proof failed because we don't know anything about $F(n-1)$.

We need to make a statement about *all* preceeding values, not just the immediate predecessor.

Let's do "strong induction"!

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

Then $F(n) = n \mod 2$ for all $n \in \mathbb{Z}^+$.

Question: What should we assume?

Let $P(k)$ be the assertion that:

- $F(k) = k \mod 2$

and let us assume that $P(1), P(2), P(3), \ldots, P(n)$ are all true!

Can we work with this?

Using strong induction

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

Then $F(n) = n \mod 2$ for all $n \in \mathbb{Z}^+$.

Let $P(k)$ be the assertion:

- $F(k) = k \mod 2$

and let us assume that $P(1), P(2), \ldots, P(n)$ are all true.
We need two base cases $n = 1$ and $n = 2$.
Does this work? (YES)
Why do we need two base cases? (You'll see later)

Using strong induction

Let $F : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

- $F(1) = 1$ and $F(2) = 0$
- $F(n) = F(n-2)$ if $n > 2$

Then $F(n) = n \mod 2$ for all $n \in \mathbb{Z}^+$.

Let $P(k)$ be the assertion:

- $F(k) = k \mod 2$

Now we want to show that for any arbitrary $n \geq 1$,

$$P(1) \wedge P(2) \wedge \ldots \wedge P(n) \to P(n+1)$$

In other words, we wish to prove that

- IF $\forall k \in \{1, 2, \ldots, n\}, F(k) = k \mod 2$
- THEN $F(n+1) = n+1 \mod 2$

Continuing this proof

Let $n \geq 2$ and assume $P(k)$ true for all $k = 1, 2, \ldots, n$.

We wish to infer that $P(n+1)$ is true.

Since $n \geq 2$, then $n + 1 \geq 3$.

Hence, by the definition of F:

$$F(n+1) = F(n-1)$$

But $n - 1 \leq n$ and so by the Inductive Hypothesis,

$$F(n-1) = n - 1 \mod 2$$

Also note that

$$n - 1 \mod 2 = n + 1 \mod 2$$

Hence

$$F(n+1) = n + 1 \mod 2$$

Finishing the proof

We showed that

$$P(1) \wedge P(2) \wedge \ldots P(n) \to P(n+1)$$

whenever $n \geq 2$.

We also had established that $P(1)$ and $P(2)$ were true.

Hence, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

In other words, we have shown that $F(n) = n \mod 2$ for all $n \in \mathbb{Z}^+$.

Q.E.D.

Why did we need two base cases?

Recall that we checked base cases $n = 1$ and $n = 2$.

We needed this to be sure that we covered every possible $n \in \mathbb{Z}^+$.

Suppose we only verified that $P(1)$ is true.

Then how would we be able to derive that $P(2)$ is true?

What we learned

We learned:

- The base case is sometimes more than one value.
- Sometimes strong induction is needed!

Proofs by induction that go wrong

Is this true?

> All finite simple graphs with minimum degree at least 1 are connected

Proofs by induction that go wrong

Is this true?

> All finite simple graphs with minimum degree at least 1 are connected

Answer:

> Consider the graph $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{(a, b), (c, d)\}$

Proofs by induction that go wrong

What's wrong with this proof of this "theorem"?

Theorem: All finite simple graphs with minimum degree at least 1 are connected

Proof: We prove by induction on n, the number of vertices.

Our Inductive Hypothesis is $P(n)$:

- all simple graphs with n vertices with minimum degree 1 or more are connected.

The base case is $n = 1$ and is vacuously true.

Let $N \geq 1$ be arbitrary, and assume $P(N)$ is true.

Proofs by induction that go wrong

Let $G = (V, E)$ be a simple graph with N vertices with minimum degree at least 1.

By the inductive hypothesis, G is connected.

Let G' be the graph formed by adding one vertex, x, to G.

If G' has minimum degree at least 1, then x must be adjacent to some vertex y.

Since y is connected (via a path) by every other vertex in G, x is also connected (via a path) to every other vertex in G'.

Hence G' is connected.

Since G was arbitrary, this proves that all simple finite graphs with minimum degree 1 are connected.