CS173
Introduction to Induction

Tandy Warnow

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Challenge questions

If you already about proofs by induction, do one of these during the class time.
We will spend the last 5-10 minutes of class doing these at the board.

1. Prove, using induction, that every positive integer at least 2 has a factorization into primes.
2. Prove, using induction, that every finite simple graph has an even number of vertices of odd degree.
3. Prove, using induction, that the n^{th} Fibonacci number $F(n)$ is at least n, where $F(0) = F(1) = 1$ and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$.

Material

- Assigned readings in Rosen, etc.
- Review of Distributive Property
- Review of nested quantifiers
- Induction proofs

Assigned Reading

Assigned reading so far:

- Various campus policies and course policies
- All PDFs for course presentations
- Rosen, pages 1-10, 25-31, 86-87 (Week 1)
- Rosen, pages 36-49, 115-120, 311-320, 344-346, 349 (Week 2)
- Rosen, list of symbols inside front cover (Week 2)

Please do the assigned reading before attending class!

Distributive property

Covered in Week 1 assigned reading (see Table 6, page 27):

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Review of nested quantifiers

Recall:

- $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ such that $x < y$
- $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}\ x < y$

These are different statements – and one of them is not true!

Sometimes these statements are written without the "such that" (and they don't change meaning!).

Induction Proofs

The *idea* behind induction is simple.

If I have a very contagious cold and I sneeze, then the person to my right will catch the cold.

Then she will sneeze and the person to her right will catch it.

Then he will sneeze, etc.

Eventually everyone (to my right) will catch the cold.

Very easy induction proof

Suppose $f : \mathbb{N} \to \{0, 1\}$ is defined by:

- $f(0) = 0$
- $f(n) = f(n-1)$ if $n > 0$

We wish to prove $f(n) = 0$ for all $n \in \mathbb{N}$.
Let $P(n)$ be the statement "$f(n) = 0$"
So we want to prove that $P(n)$ is true for all $n = 0, 1, 2, \ldots$.

Very easy induction proof

A proof by induction has two steps:

1. Confirming that $P(n)$ is true for some initial values
2. Check that when $P(n)$ is true it implies that $P(n+1)$ is true.

The first step is called the “Base Case” and the second step is called the “Induction Step”.

The Induction Step has two parts: The Inductive Hypothesis (assuming $P(n)$ is true for some arbitrary n) and then using that to deduce $P(n+1)$ is true.

Very easy induction proof

Suppose $f : \mathbb{N} \to \{0, 1\}$ is defined by:

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We wish to prove $f(n) = 0$ for all $n \in \mathbb{N}$.
Let $P(n)$ be the statement "$f(n) = 0$"
So we want to prove that $P(n)$ is true for all $n = 0, 1, 2, \ldots$.

Very easy induction proof

Our base case is $n = 0$, and asserts that $f(0) = 0$, which is true.

Very easy induction proof

Let $N \geq 0$ be arbitrary.

Assume $P(N)$ is true. Hence, $f(N) = 0$.

Since $N \geq 0, N + 1 \geq 1$.
Hence, by definition, $f(N + 1) = f(N) = 0$.
In other words, we have shown:

$$P(N) \rightarrow P(N + 1)$$

Since N was arbitrary, $P(N)$ is true for all N.

In other words, we have shown that $f(n) = 0$ for all $n = 0, 1, 2, \ldots$.

Necessary parts of induction proofs

In the usual induction proof, you want to prove that some property $P(n)$ is true for all $n \in N, n \geq n_0$.

Sometimes you aren't really told what $P(n)$ is – and you will need to figure this out.

How to approach a proof by induction

- Let $\mathbb{N}^{\geq n_0} = \{n \in \mathbb{N} | n \geq n_0\}$. Find a way of defining a logical statement $P(n)$ (that depends on a parameter n) so that what you want to prove is:

$$\forall n \in \mathbb{N}^{\geq n_0}, P(n)$$

- Prove the base case: show $P(n_0)$ is true
- Say "Let N be arbitrary".
- State the Inductive Hypothesis: "$P(N)$ is true"
- Show that $P(N) \to P(N+1)$
- Point out that N was arbitrary so the result holds for all $N \geq n_0$.
- Optional: say "Q.E.D."

The Inductive Hypothesis

The inductive hypothesis must be a statement of the form $P(N)$, where N is arbitrary.
Example of bad inductive hypotheses:

- The inductive hypothesis is that $g(N) > n$ for all N (Bad because you are asserting what you want to prove)
- The inductive hypothesis is $N^2 + 3$ (Bad because you need to assert something that is either true or false)
- The inductive hypothesis is that $f(3) = 17$ (Bad because $P(N)$ needs to depend on the parameter N).

The base case

The base case is the first value n_0 for which you want to prove the statement true.
Often $n_0 = 1$, but not always. Be careful to check.
Sometimes you need to establish several base cases.
Typically the base case is done properly.
But sometimes it's missing!

Class Exercise

Let $f(n)$ be defined by

- $f(0) = 3$
- $f(n) = f(n-1) + 1$ if $n \geq 1$

Find a closed form solution to $f(n)$ and prove it true by induction on n.

Another simple proof by induction

Let A_n for $n \in \mathbb{Z}^+$ be a set defined by:

- $A_1 = \emptyset$
- $A_n = A_{n-1} \cup \{n-1\}$

We wish to prove that $A_n = \{1, 2, \ldots, n-1\}$ for all $n \in \mathbb{Z}^{\geq 2}$.

Let's prove this by induction!

Base case is $n = 2$.
By definition, $A_2 = A_1 \cup \{1\} = \emptyset \cup \{1\} = \{1\}$.
So the base case is good!

The Inductive Hypothesis is the statement
$P(n) =$ "$A_n = \{1, 2, \ldots, n-1\}$"

Let $n \geq 2$ be arbitrary.
Can we show $P(n) \rightarrow P(n+1)$?

Showing $P(n) \to P(n+1)$

Recall $A_n = A_{n-1} \cup \{n-1\}$
The Inductive Hypothesis is the statement
$P(n) =$ "$A_n = \{1, 2, \ldots, n-1\}$"

Let $n \geq 2$ be arbitrary.

Let $n \geq 2$.
By definition, $A_{n+1} = A_n \cup \{n\}$.
Since $n \geq 2$, by the I.H., $A_n = \{1, 2, \ldots, n-1\}$.
Hence, $A_{n+1} = \{1, 2, \ldots, n\}$, which is $P(n+1)$.

Hence we showed $P(n) \to P(n+1)$.

Q.E.D.

Recurrence relations

Recurrence relations are generally functions or sets that are defined recursively, as in

1. $g(1) = 3$ and $g(n) = 3 + g(n-1)$ for $n \geq 2$
2. $A(1) = \{f : \mathbb{Z}^+ \to \mathbb{Z} | f(1) = 1\}$ and
   $A(n) = A(n-1) \cup \{f : \mathbb{Z}^+ \to \mathbb{Z} | f(n) = 1\}$ for $n \geq 2$
3. $B(1) = \{f : \mathbb{Z}^+ \to \mathbb{Z} | f(1) = 1\}$ and
   $B(n) = B(n-1) \cap \{f : \mathbb{Z}^+ \to \mathbb{Z} | f(n) = 1\}$ for $n \geq 2$
4. $C(1) = \mathbf{T}$ (i.e., True) and $C(n) = \neg C(n-1)$ if $n \geq 2$.

Questions:

- Can you find closed forms for any of these?
- Can you prove your closed form correct by induction?

Summary

We learned:

- Induction is a powerful technique for proving theorems
- Induction proofs are not that difficult!
  - Write down what you want to prove as a statement $P(n)$ that depends on a parameter n
  - Show that $P(n)$ is true for some base case(s)
  - Show that $P(n) \rightarrow P(n+1)$