CS173 Lecture B, September 8, 2015

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Today's lecture

- We will finish induction proofs
- We will begin material on sets

Another induction proof

Let $f : NxN \rightarrow N$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

We would like to prove that $f(n, m) \geq n + m$ for all $n \geq 1$ and $m \geq 1$.

Note, when $n = 1$ or $m = 1$, the statement is true. What is our inductive hypothesis?

Another induction proof, continued

Recall that

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

Our inductive hyopthesis will be:

> $\exists K \geq 2$ such that $f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$

The base case is $K = 2$; hence $n = m = 1$ and the statement holds.
Now assume that for some arbitrary K, the statement holds.
We wish to show it holds also for $K + 1$.

Another induction proof, continued

Let n, m be given so that $n + m = K + 1$.
We have to do a case analysis;
Case 1: If $n = 1$ or $m = 1$, then $f(n, m) = n + m$, and the statement holds.
Case 2: Both n and m are at least 2. Then, by definition $f(n, m) = f(n-1, m) + f(n, m-1)$.

Since $n+m-1 = K$, we can apply the inductive hypothesis; hence:

$$f(n-1, m) \geq (n-1) + m = n + m - 1$$

and

$$f(n, m-1) \geq n + (m-1)$$

Hence,

$$f(n, m) \geq 2(n + m - 1) = 2n + 2m - 2 > n + m - 1$$

Since K was arbitrary, the statement holds for all K. Q.E.D.

What we learned

We learned:

- The base case is sometimes more than one value.
- The inductive hypothesis is sometimes not on an obvious parameter, but on something built using obvious parameters (like the sum)
- Induction can be used to prove upper or lower bounds on recursively defined functions.
- Induction proofs are not difficult!

Two-person games

Remember the original two person game?
Who wins?
Prove your assertion true by induction on the total number of rocks in the game.

Introduction to sets

- Sets are collections of objects, where each object appears at most once. These objects can be numbers, strings, graphs, functions, or even sets. Thus, sets are quite general.
- Sets can be described (and defined) in several ways, including *set builder notation*.
- The empty set contains nothing! It is written as $\emptyset$.
- Sets can be defined recursively. Therefore, you can prove theorems about sets using induction.
- There are finite sets and also infinite sets. There are countable sets and uncountable sets.
- You can take unions of sets, intersections of sets, and compute the differences between sets.
- Venn Diagrams help you visualize these combinations of sets.

Describing a set

Consider the set of integers between 3 and 10. We can write this in several ways; the following are just some of these ways!

- $\{3, 4, 5, 6, 7, 8, 9, 10\}$
- $\{10, 3, 5, 6, 9, 4, 7, 8\}$
- $\{x \in Z : 10 \geq x \geq 3\}$
- $\{x \in Z : 2 < x < 11\}$

The last two ways of defining the set are using "set builder notation".

Element of and Subset of

Let A be a set.

- We write "$X \in A$" if X is one of the elements of A.
- We say "X is a **subset** of A" (written $X \subseteq A$) if X is a *set* and every element of X is an element of A. Note, we can also write this as $A \supseteq X$, and say that "A is a **superset** of X".
- We say that "X is a **proper subset** of A" if X is a subset of A but $X \neq A$. This is denoted by $X \subset A$ (or as $A \supset X$). Sometimes we write this as $X \subsetneq A$ (or as $A \supsetneq X$). We express this also as "A is a proper superset of X."

Element of and Subset of

Let $A = \{1, 5, 7, 10, 12\}$. Which of the following statements are true?

- $1 \in A$
- $\{1, 3\} \in A$
- $\{1, 5\} \in A$
- $\{1, 5\} \subset A$
- $A \subset \{1, 5\}$
- $\emptyset \in A$
- $\emptyset \subset A$

Venn Diagrams

Venn Diagrams are useful ways of visualizing sets and their intersections; see http://mathworld.wolfram.com/VennDiagram.html

Venn Diagrams

Basic operations

Let A and B be sets. Then

- $A \cup B = \{x : x \in A \vee x \in B\}$
- $A \cap B = \{x : x \in A \wedge x \in B\}$
- $A \setminus B = \{x \in A : x \notin B\}$. Sometimes this is written as $A - B$.
- $A \triangle B = (A \cup B) \setminus (A \cap B)$
- For the complement of a set to be defined, we need to know the **universe** (often denoted by U). Then $A^c = U \setminus A$.

Visualizing these operations

De Morgan's laws

Let U denote the universe, and A and B subsets of U. Then

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

$$(A \cap B)^c = A^c \cup B^c$$

Inclusion-Exclusion

Question: What is $|A \cup B|$?

Inclusion-Exclusion

Question: What is $|A \cup B|$?
Answer: $|A \cup B| = |A| + |B| - |A \cap B|$
Proof: Venn Diagram

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion

Question: What is $|A \cup B \cup C|$?
Answer: $|A \cup B \cup C| =$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

$$+|A \cap B \cap C|$$

Proof: Venn Diagram

Inclusion-Exclusion formula for $|A \cup B \cup C|$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion formula for $|A \cup B \cup C \cup D|$?

Applying Inclusion-Exclusion

Question: How many numbers are there between 1 and 1000 that are divisible by 3 or by 7?
Answer: Apply Inclusion-Exclusion.

How many numbers between 1 and 1000 are divisible by 3 or 7?

Let $A = \{x \in Z^+ : 1 \leq x \leq 1000 \wedge 3|x\}$
Let $B = \{x \in Z^+ : 1 \leq x \leq 1000 \wedge 7|x\}$
We want to know $|A \cup B|$.

- By Inclusion-Exclusion, $|A \cup B| = |A| + |B| - |A \cap B|$
- $A = \{3, 6, 9, \ldots, 999\}$, so $|A| = 333$
- $B = \{7, 14, 21, \ldots, 987\}$, so $|B| = 141$
- $A \cap B = \{21, 42, \ldots, 987\}$, so $|A \cap B| = 47$

Hence, $|A \cup B| = 333 + 141 - 47 = 427$

Different types of sets

Sets don't have to only be of numbers. They can also be of any kind of object (graphs, functions, sets, alphanumeric strings, etc.). Examples:

- The set $\{f : R \to R\}$ (i.e., the set of functions from the real numbers to the reals).
- The set $\{f : R \to R : \forall x \in Z, f(x) = x\}$ (the set of functions from the reals to the reals that act as the identity function on the integers)
- The set Σ^*, where Σ is an alphabet, is the set of all finite length strings (including the empty string, λ) over Σ
- The set Σ^+, where Σ is an alphabet, is the set of finite length strings - not including the empty string - over Σ
- $\mathcal{P}(X)$, the set of all subsets of X (called the power set of X). Note $\emptyset \in \mathcal{P}(X)$ for all X.

A recursively defined set

Let the sets $A_1, A_2, \ldots, A_n$ be defined by:

- $A_1 = \{3\}$
- $A_n = \{x : \exists y \in A_{n-1}, x = y + 1\}$, for $n > 1$

Questions:

- What is A_2?
- What is A_3?
- Find a closed form solution for A_n and prove your formula correct by induction on n.
- What is $\bigcup_i A_i$? Prove your statement true.

Proving $X \subsetneq Y$

Sometimes you will want to prove that a set $X \subsetneq Y$ (i.e., X is a proper subset of Y).
To do this, you need to:

- Prove $X \subseteq Y$
- Prove $X \neq Y$

To prove the second statement, you will need to find $y \in Y \setminus X$.

Proving $X \subsetneq Y$

Let

- $A = \{x \in Z : \exists y \in Z \text{ s.t. } x = 2y\}$
- $B = \{x \in Z : \exists y \in Z \text{ s.t. } x = 6y\}$

Which of the following are true?

- $A = B$
- $A \subseteq B$
- $B \subseteq A$

Comparing two sets

Recall:

- $A = \{x \in Z : \exists y \in Z \text{ s.t. } x = 2y\}$
- $B = \{x \in Z : \exists y \in Z \text{ s.t. } x = 6y\}$

Note $2 \in A \setminus B$, so $A \neq B$.
Question: Is B a subset of A?

Comparing two sets

$A = \{x \in Z : \exists y \in Z \text{ s.t. } x = 2y\}$
$B = \{x \in Z : \exists y \in Z \text{ s.t. } x = 6y\}$
We will prove $B \subseteq A$.
Proof: We will show that every element of B is an element of A.

- Pick arbitrary $b \in B$.
- By definition, $\exists x \in Z$ such that $b = 6x$.
- Let $y = 3x$. Then $y \in Z$, and also $2y \in Z$.
- Note that $b = 2y$.
- Hence, by definition $b \in A$.
- Since b was arbitrary, this proves that $B \subseteq A$.

Since $B \neq A$ and $B \subseteq A$, it follows that B is a proper subset of A, i.e.,

$$B \subsetneq A$$

Proving two sets are equal

To prove two sets A and B are equal, you need to:

- Prove $A \subseteq B$
- Prove $B \subseteq A$

Do not forget to do both directions!

Another recursively defined set

- $A_0 = \{\{0\}\}$.
- $A_n = A_{n-1} \cup \{B \cup \{n\} : B \in A_{n-1}\}$ for $n \geq 1$.

Note that for all i, $A_i \subset \mathcal{P}(Z)$.

- What is $|A_0|$?
- Is $A_0 \subseteq A_1$?
- Show one element of A_1.
- Show another element of A_1.
- What is $|A_1|$?
- Is $A_1 \subseteq A_2$?
- Give one element of $A_2 \setminus A_1$.
- Is $A_n \subseteq \mathcal{P}(\{0, 1, \ldots, n\})$?
- Is $A_n = \mathcal{P}(\{0, 1, \ldots, n\})$?
- What is $\bigcup_{i=0} A_i$? (infinite union)

Finite sets and cardinality

A **finite set** is a set where you can write all the elements down explicitly. For example, $A = \{1, 3, 4, 9\}$ is finite.

Given a finite set A, we write $|A|$ to denote the number of elements of A. Thus, $|A| = 4$ for the set above.

What is $|\emptyset|$?

Bijections

A **bijection** from a set A to a set B is a function $f : A \to B$ such that

- f is 1-1 (so that $f(a) = f(b) \Rightarrow a = b$)
- f is *onto* (so that $\forall b \in B, \exists a \in A$ such that $f(a) = b$)

Note that every set has a bijection from itself to itself (the identity function).
Question: $\exists X, Y$ such that $Y \subsetneq X$ and $f : X \to Y$ with f a bijection?

Infinite sets

An infinite set is a set that isn't finite.

Since we don't have a very nice definition of "finite", a better definition is that a set A is *infinite* if there is a *proper subset* $B \subset A$ and there is a **bijection** (function that is 1-1 and onto) between A and B.
Example:

- Z^+ is infinite because there is a bijection from Z^+ to $A = \{2, 3, 4, \ldots\}$! Just let $f(n) = n + 1$; then $f : Z^+ \to A$, and f is $1 - 1$ and onto.
- $A = \{1, 3, 4, 9\}$ is not infinite; there is no proper subset B of A for which A and B have a bijection.

Can you show that the even integers are infinite?

Countable and uncountable sets

A countable set can either be finite, or countably infinite. X is a countably infinite set *iff* $\exists f : Z^+ \rightarrow X$ s.t. f is a bijection.

Otherwise, an infinite set X that is not countably infinite is said to be *uncountable*.

Questions:

- Is the set of rational numbers countable?
- Is the set of negative integers countable?
- Is the set of integers countable?
- Is the set of real numbers countable?
- Is the set $\{f : Z \rightarrow \{0, 1\}\}$ countable?
- Is the set $\{f : \{0, 1\} \rightarrow Z\}$ countable?
- Is the set $\{f : Z \rightarrow \{0\}\}$ countable?