CS173 Lecture B, September 29, 2015

Tandy Warnow

September 30, 2015

CS 173, Lecture B
September 29, 2015
Tandy Warnow

Piazza

It seems that only a bit more than half of you are enrolled in Piazza.
Please enroll - I am using it to send important messages.
The passcode is...
(The goddess of wisdom...from the Greeks.)
According to Wikipedia, she's also the goddess of wisdom, courage, inspiration, civilization, law and justice, mathematics, strength, war strategy, the arts, crafts, and skill.

Today

We will cover

- Basic inductive proof
- Strong inductive hypothesis - especially when it's needed
- Proving statements about two variables using induction
- Starting to think about dynamic programming

Examlet on Thursday

You will be given a recursively defined function of two variables, and asked to compute some values, and then prove a theorem about the function.

You will need to prove the theorem by induction.

Since the function is on two variables, you'll need to figure out how to do the induction proof.

The only hard part really is the inductive hypothesis. So practice this.

Induction Proofs

The *idea* behind induction is simple. If I have a very contagious cold, and I sneeze then the person to my right will catch the cold; then she will sneeze, and the person to her right will catch it; and then he will sneeze, etc. Eventually everyone (to my right) will catch the cold.

Necessary parts of induction proofs

- Inductive Hypothesis
- Base case
- Use the inductive hypothesis (that a statement is true for some value K) to prove it true for the next value $(K+1)$.
- Point out that K was arbitrary so the result holds for all K.
- Optional: say "Q.E.D."

The Inductive Hypothesis

The inductive hypothesis must be a statement that is either true or false, and that depends on a parameter n.

Your inductive hypothesis must be that the statement is true for all values n between some base case n_0 and some specific (but arbitrary) value K.

It is best if you write the inductive hypothesis very carefully - half the points are based on this.

The usual inductive hypothesis is of the form

$$\exists K \in Z^+ \text{ such that } \forall n \in \{n_0, n_0 + 1, \ldots, K\}, P(n) \text{ is true}$$

Note that K is quantified (by $\exists K$).

The worst kind of inductive hypothesis is a statement of what you want to prove. Make sure you don't do that.

The base case

The base case is the first value(s) for which you want to prove the statement true.
Often $n_0 = 1$, but not always. Be careful to check. Sometimes you need to establish several base cases.
Typically the base case is done properly. But sometimes it's missing - and that's a big mistake.

Recurrence relations

Recurrence relations are generally functions defined recursively, as in

1. $g(1) = 3$ and $g(n) = 3 + g(n-1)$ for $n \geq 2$
2. $f(1) = f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$.

Induction gives you a way of proving closed form solutions, or upper and lower bounds, on recursively defined functions (i.e., recurrence relations).

Note that $g(n)$ depends only on $g(n-1)$, but $f(n)$ depends on $f(n-1)$ and $f(n-2)$. Hence you *must* use strong induction for anything you want to prove about $f(n)$, but you *could* have used weak induction for $g(n)$.

Strong induction is always valid, so practice using it.

Using induction to prove theorems about functions of two variables

Let $f : Z^+ \times Z^+ \rightarrow Z^+$ be defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

We would like to prove that $f(n, m) \geq n + m$ for all $n \geq 1$ and $m \geq 1$.

Base case: $n = 1$ or $m = 1$ follows immediately. So we prove the rest by induction.
What is our inductive hypothesis?

Inductive hypothesis

Recall that

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

We want a single parameter that always "goes down". Note we can't do induction on n because $f(n, m)$ depends on $f(n, m-1)$.
We can't do induction on m because $f(n, m)$ depends on $f(n-1, m)$.
We need something that goes down... so that $f(n, m)$ depends on values *to which the inductive hypothesis can be applied.*
What goes down?

Inductive hypothesis, continued

The sum of the parameters goes down! So, our inductive hyopthesis will be:

> $\exists K \geq 2$ such that $f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$

Note, you could rewrite this in many ways, all of them valid. Here's another:

- $\exists K \geq 2$ such that $\forall$ positive integers n, m s.t. $\;n + m \leq K$, $f(n, m) \geq n + m$

In fact, the second way of writing the I.H. is a bit nicer.

The base case

Recall $f : Z^+ \times Z^+ \rightarrow Z^+$, and the inductive hypothesis is:

> $\exists K \geq 2$ such that $f(n,m) \geq n+m$ for all positive integers n, m with $n+m \leq K$

The smallest value for $n+m$ is 2; hence, the base case is $K = 2$.
When $K = 2$, $n = m = 1$ and the statement holds.
Now assume that for some arbitrary K, the statement holds. We wish to show it holds also for $K+1$. In other words, we want to show:

$$\forall n, m \text{ such that } n+m = K+1, f(n,m) \geq n+m$$

Another induction proof, continued

Let n, m be given so that $n + m = K + 1$. If $n = 1$ or $m = 1$, then $f(n, m) = n + m$, and the statement holds.
So assume $n \geq 2$ and $m \geq 2$, so that

$$f(n, m) = f(n-1, m) + f(n, m-1)$$

Note that $n + m = K + 1$ and so $n + m - 1 = K$. Hence we can apply the Inductive Hypothesis to $f(n-1, m)$ and $f(n, m-1)$.
Therefore,

$$f(n-1, m) \geq n + m - 1$$

and

$$f(n, m-1) \geq n + m - 1$$

Hence

$$f(n, m) = f(n-1, m) + f(n, m-1) \geq 2(n + m - 1)$$

Another induction proof, continued

So far we have shown that when n, m are both at least 2 and $n + m = K + 1$, then

$$f(n, m) = f(n-1, m) + f(n, m-1) \geq 2(n + m - 1)$$

Note that n, m are both at least 2, and so $n + m - 1 \geq 3$. Hence,

$$2(n + m - 1) > n + m - 1$$

and therefore

$$2(n + m - 1) \geq n + m$$

Putting this all together, we obtain

$$f(n, m) = f(n-1, m) + f(n, m-1) \geq 2(n + m - 1) \geq n + m$$

which is what we wished to prove.
Since K was arbitrary, the statement holds for all K. Q.E.D.

Summarizing what we did

Recall that we had a recursively defined function $f : Z^+ \times Z^+ \to Z^+$, defined by

- $f(n, m) = n + m$ if $n = 1$ or $m = 1$,
- $f(n, m) = f(n-1, m) + f(n, m-1)$, otherwise

We wanted to prove that $f(m, n) \geq m + n$ for all positive integers m, n.

It is easy to verify this inequality for the case where $m = 1$ or $n = 1$. To prove it true for all m, n, we used induction.

But induction must be done for some single parameter.

We used $K = m + n$ as our single parameter.

Our inductive hyopthesis was

> $\exists K \geq 2$ such that $f(n, m) \geq n + m$ for all positive integers n, m with $n + m \leq K$