CS 173 Lecture B, September 22, 2015

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Relations

Associated reading from textbook: Chapter 6. However, I'll present this somewhat differently.

Binary relations are sets of ordered pairs, expressing relationship (of some sort). Thus, a binary relation R on a set S is a subset of $S \times S$.

Some examples of binary relations

Consider the set R of ordered pairs of integers where $(x, y) \in R$ if and only if x divides y.
Then $(3, 6) \in R$, and $(2, 2) \in R$, but $(6, 3) \notin R$. Thus, the order matters!
Questions:

- Is there an integer x such that $(x, y) \in R$ for all $y \in Z$?
- Is there an integer p such that $(n, p) \notin R$ for all $n \in Z$?
- Is there an integer p such that $(p, p) \notin R$?
- Is there an integer p such that $(p, 0) \in R$?
- Is there an integer p such that $(0, p) \in R$?

Other examples of binary relations

Let Z denote the set of integers. Consider the relation $R \subseteq Z \times Z$ defined by $(x, y) \in R$ if and only if $x + y = 0$.
Questions:

- Give some examples of elements of R.
- Is it the case that $\forall x \in Z$ and $\forall y \in Z$, $(x, y) \in R \Rightarrow (y, x) \in R$?
- Is it the case that for all integers x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$?
- Is $(x, x) \in R$ for all $x \in Z$?

Another relation

Let $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and let $R \subseteq Z^+ \times Z^+$ such that $R = \{(x, y) : x \in A, y \in A, x \leq y \leq x + 2\}$.

Questions:

- Give an example of some elements of R, and at least one ordered pair $(a, b) \notin R$ where $(a, b) \in A \times A$.
- Draw a graph with vertices from the set A, and with a directed edge from $u \to v$ if and only if $(u, v) \in R$. What is the maximum indegree of any node? Maximum outdegree?
- Does this relation R have $(x, x) \in R$ for all $x \in A$?
- Is this relation **symmetric**? In other words, does $(x, y) \in R$ imply that $(y, x) \in R$?
- Is this relation **anti-symmetric**? In other words, if $(x, y) \in R$ and $(y, x) \in R$, does it follow that $x = y$?
- Is this relation **transitive**? In other words, if $(x, y) \in R$ and $(y, z) \in R$, does it follow that $(x, z) \in R$?

More on binary relations

Properties of binary relations R on a set S that are often interesting.

- Reflexive: $\forall x \in S, (x, x) \in R$
- Irreflexive: $\forall x \in S, (x, x) \notin R$
- Symmetric: $\forall x \in S, \forall y \in S, (x, y) \in R \Rightarrow (y, x) \in R$
- Anti-symmetric: $\forall x \in S$ and and $\forall y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.
- Transitive: $\forall x \in S, y \in S, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$

Relations on sets that are not numbers

Binary relations are subsets of $A \times A$ for some set A. So far we have talked only about relations that are subsets of $A \times A$ where $A \subseteq Z$. But this is not the only type of binary relation to consider. Let A denote the set of people in the USA. Consider the following relations on this set:

- Suppose $(a, a') \in R$ if and only if a and a' have the same birthday (not considering the year). What properties does this relation satisfy?
- Suppose $(a, a') \in R$ if and only if a and a' have the same father.
- Suppose $(a, a') \in R$ if and only if a is older than a'.
- Suppose $(a, a') \in R$ if and only if a and a' live in the same city on this date.
- Suppose $(a, a') \in R$ if and only if a doesn't like a'.

What properties do each of these relations satisfy? (Symmetric, anti-symmetric, reflexive, irreflexive, or transitive?)

Partial order

A partial order is a binary relation that is reflexive, anti-symmetric, and transitive. Standard example:

- R contains pairs (x, y) if and only if x divides y (for integers x and y)

Linear order

A linear order is a partial order where all pairs of elements are comparable. In other words, a linear order a partial order a set A such that $\forall \{x, y\} \subseteq A$, $(x, y) \in R$ or $(y, x) \in R$. Standard examples

- R contains pairs (x, y) if and only if $x < y$ (for real numbers x and y)
- R contains pairs of people (x, y) if and only if x is older than y

Equivalence Relations

An equivalence relation is a binary relation that is based on "equivalence classes". Think of

- R contains all pairs of people with the same birthday
- R contains all pairs of integers with the same parity
- R contains all pairs of integers with the exact same set of prime factors
- R contains all pairs of graphs with the same number of vertices
- R contains all pairs of functions that map the positive integers to the positive integers that have the same value on input 1.

Equivalence Relations

More formally, an equivalence relation is a binary relation on set A that is reflexive, symmetric, and transitive.

It is often easier to define an equivalence relation by its equivalence classes!

Questions

For each of the following relations, (a) prove each is an equivalence relation, (b) give an example of an element of the relation, and (c) determine the number of equivalence classes it has.

- R contains all pairs of people in the USA with the same birthday
- R contains all pairs of positive integers with the same parity
- R contains all pairs of positive integers between 2 and 100 that have the same set of prime factors
- R contains all pairs of finite graphs with the same number of vertices
- R contains all pairs of functions that map the set $\{1, 2, \dots, 100\}$ to $\{1, 2, 3, 4, 5\}$ that have the same value on input 1.

Questions

R contains all pairs of people in the USA with the same birthday
We prove R is an equivalence relation.

- Let A be the set of people in the USA, and let $a \in A$ be arbitrary. Note that a has the same birthday as a, and so $(a, a) \in R$. Hence R is reflexive.
- Now let a and b be arbitrary people in the USA such that $(a, b) \in R$. Hence a and b have the same birthday. Hence $(b, a) \in R$ as well. Hence R is symmetric.
- Now suppose a, b, c are three arbitrary people in the USA, and that $(a, b) \in R$ and $(b, c) \in R$. Hence a and b have the same birthday, and b and c have the same birthday. Hence a and c have the same birthday, and so $(a, c) \in R$. Hence T is transitive.

Hence R is an equivalence relation. There are 366 equivalence classes, one for each of the possible birthdays.

Equivalence relations and equivalence classes

- R contains all pairs of positive integers with the same parity. *The equivalence classes are* $\{1, 3, 5, \ldots\}$ *and* $\{2, 4, 6, 8, \ldots\}$*.*
- R contains all pairs of positive integers between 2 and 14 that have the same set of prime factors. *The equivalence classes are defined by the set of prime factors. So the equivalence classes are* $\{2, 4, 8\}, \{3, 9\}, \{6, 12\}, \{5\}, \{7\}, \{10\}, \{11\}, \{13\}$, *and* $\{14\}$*.*
- R contains all pairs of finite graphs with the same number of vertices *The equivalence classes are defined by the number of vertices. So an example of an equivalence class is all graphs with exactly two vertices. There is an infinite (but countable) number of equivalence classes.*
- R contains all pairs of functions that map the set $\{1, 2, \ldots, 100\}$ to $\{1, 2, 3, 4, 5\}$ that have the same value on input 1. *There are five equivalence classes, defined by the value of the function on input* 1*.*

Functions

A function $f : A \rightarrow B$ is actually also just a binary relation, but one with some properties.
To describe f as a binary relation (which we will call R_f), we write it as a relation on $A \cup B$. Then, $(a, b) \in R_f$ if $f(a) = b$.
Note that a function has some required properties:

- $\forall a \in A, \exists b \in B$ such that $(a, b) \in R_f$.
- $\forall a \in A, \forall b \in B, b' \in B$, if $(a, b) \in R_f$ and $(a, b') \in R_f$ then $b = b'$

Note that this definition of a function as a set of ordered pairs does *not* specify the domain and co-domain...

Drawing binary relations as graphs

When a binary relation R on a set A is finite, it is often easy and helpful to visualize it as a directed graph.

The nodes of the graph are the elements of A, and the elements of R are represented by directed edges. Thus, the element (x, y) is represented by the edge from x to y.

Note that this kind of graph can have self-loops (edges $x \longrightarrow x$) but not parallel edges (two or more directed edges from x to y).

It is easy to check whether a relation is **reflexive** or **irreflexive**, **symmetric** or **anti-symmetric**, and even **transitive**, when you have its visualization as a graph.

Visualizing functions as directed bipartite graphs

Given a function $F : X \to Y$, if X is finite, we can visualize F as a directed bipartite graph.
The left hand side are the vertices for X, and the right hand side are the vertices in Y.
There is a directed edge from x to $F(x)$ for each $x \in X$.
Note: There must be exactly one edge leaving each $x \in X$ for this graph to represent a function from X to Y.
Questions:

- What will the graph look like when F is $1 - 1$?
- What will the graph look like when F is onto?

Examlet #3: Thursday, September 24

Two questions:

1. Define the various properties of a binary relation (i.e., reflexive, irreflexive, symmetric, anti-symmetric, and transitive) using proper mathematical language
2. Take a definition of a binary relation R on a set S and prove (or disprove) each of the various properties.

Remember: the set S may not be a set of integers. Instead, S could be a set of sets, a set of functions, a set of graphs, a set of people, a set of cities, etc.