CS 173 Lecture B, September 17, 2015

Tandy Warnow

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Class exercise

Let $F : Z^+ \to Z$ be defined recursively by

- $F(1) = 3$
- $F(n) = 2F(n-1) + 1$

Prove that $F(n) > 2^{n-1}$ for all $n \in Z^+$.
Proof by contradiction. Let $P(n) \equiv F(n) > 2^{n-1}$. Note that $P(1)$ is true ($F(1) = 3 > 2^{1-1} = 1$).
If it is not the case that $P(n)$ is true for all $n \in Z^+$, then there is at least one positive integer n for which $P(n)$ is false. Let N be the smallest such positive integer. Note $N > 1$. Hence $N - 1 \geq 1$, and so $P(N-1)$ is true.
By definition, $F(N) = 2F(N-1) + 1$. Since $P(N-1)$ is true, $F(N-1) > 2^{N-2}$. Hence

$$F(N) > 2 \times 2^{N-2} + 1 = 2^{N-1} + 1 > 2^{N-1}$$

Hence $P(N)$ is true, contradicting our assumption.

Recursively defined set problem

Let A_n, $n \in Z^+ \cup \{0\}$, be defined by $A_0 = \{0\}$, and $A_n = A_{n-1} \cup \{n^2\}$ if $n > 0$. Prove that $A_n = \{i^2 : 0 \le i \le n, i \in Z\}$ for all integers $n \ge 0$.

Proof by contradiction. Let $P(n)$ be the statement $A_n = \{i^2 : 0 \le i \le n, i \in Z\}$. Note that $P(0)$ is true.

If it is not the case that $P(n)$ is true for all non-negative integers n, then there is a smallest value N for which $P(N)$ is false. Note that $N > 0$. Hence, $N - 1 \ge 0$, and therefore $P(N-1)$ is true.

By definition, $A_N = A_{N-1} \cup \{N^2\}$.

Since $P(N-1)$ is true, $A_{N-1} = \{i^2 : 0 \le i \le N-1, i \in Z\}$.

Hence,

$$A_N = \{i^2 : 0 \le i \le N-1, i \in Z\} \cup \{N^2\}$$
$$= \{i^2 : 0 \le i \le N, i \in Z\}$$

In other words, $P(N)$ is true, contradicting our assumption.