CS173 Lecture B, September 10, 2015

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Examlet Today

Four problems:

- One induction proof
- One problem on simplifying a logical expression
- One problem where you answer whether a logical expression is a tautology, satisfiable but not a tautology, or not satisfiable
- One problem where you take a 2CNF formula, and construct the graph that represents the formula

Things to remember about logic

For A and B logical expressions (or just variables), and for T and F the logical constants **true** and **false**, respectively,

- $A \Rightarrow B \equiv \neg A \vee B$
- $T \wedge A \equiv A$
- $T \vee A \equiv T$
- $F \wedge A \equiv F$
- $F \vee A \equiv A$
- De Morgan's laws
- Distribution properties
- Contrapositive: $(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$

Simplifying logical expression

Objective: Remove all $\Rightarrow$ and parentheses.
Examples:

- $x \Rightarrow y$
  Solution: $\neg x \vee y$
- $\neg x \Rightarrow y$
  Solution: $x \vee y$
- $\neg(x \Rightarrow y)$
  Solution: $\neg(\neg x \vee y) \equiv \neg\neg x \wedge \neg y \equiv x \wedge \neg y$

Harder simplification problem from reading quiz

Simplify this expression:

$$(\neg A \Rightarrow \neg B) \Rightarrow A$$

$(\neg A \Rightarrow \neg B) \Rightarrow A$

$\equiv (A \vee \neg B) \Rightarrow A$ (because $X \rightarrow Y \equiv \neg X \vee Y$)

$\equiv \neg(A \vee \neg B) \vee A$ (same reason)

$\equiv (\neg A \wedge \neg\neg B) \vee A$ (by De Morgan's law)

$\equiv (\neg A \wedge B) \vee A$ (because $\neg\neg X \equiv X$)

$\equiv (\neg A \vee A) \wedge (B \vee A)$ (Distribution)

$\equiv T \wedge (B \vee A)$ (because $X \vee \neg X \equiv T$)

$\equiv B \vee A$ (because $T \wedge X \equiv X$)

$\equiv A \vee B$ (obvious)

Tautologies

A **tautology** is a logical expression that is always true. Hence if you simplify it, it ends up as **T**. Example:

$$A \to A$$

To prove that a logical expression is a tautology, you can do one of the following:

- Construct a truth table and show all settings for the variables produce **T** for the expresion
- Simplify the expression and show is equivalent to **T**

To prove that a logical expression is *not* a tautology, you should find a setting for the variables that makes the expression false. For example: $A \wedge B$ is not a tautology, because setting $A = T$ and $B = F$ makes it false.

Satisfiability

A logical expression is satisfiable if $\exists$ setting of the boolean variables that makes the expression true. Tautologies are obviously satisfiable. But some expressions are satisfiable and are not Example:

$$A \Rightarrow B$$

Setting $A = T$ and $B = F$ makes the expression false, while setting $A = B = F$ makes the expression true.

Summary so far

- To prove that a logical expression is a tautology, you should show it reduces to **T**, or else produce the truth table and show all settings make the expression true.
- To prove that a logical expression is satisfiable, just find one setting of the variables that makes the expression true.
- To prove that a logical expression is never satisfiable, you should show that it reduces to **F**, or else produce the truth table and show all settings make the expression false.
- To show that a logical expression is satisfiable but not a tautology, show one setting of the variables that makes it true, and another setting that makes it false.

The graph representation for 2SAT

2SAT is the problem where the input is a 2CNF formula, and you wish to know if the formula is satisfiable.
The graph algorithm constructed a graph with

- One vertex for each literal (X and $\neg X$)
- Two directed edges for each clause

So:

$$A \vee B$$

is represented by two edges:

$$\neg A \Rightarrow B$$

and

$$\neg B \Rightarrow A$$

The graph algorithm for 2SAT

The algorithm checks for a directed cycle that includes X and $\neg X$ for some variable X. If such a cycle exists, then the 2CNF formula is not satisfiable; otherwise it is satisfiable.
Note: a directed path from X to $\neg X$ just means that X *must be false*. Similarly, a directed path from $\neg X$ to X means that X *must be true*.

Example

Consider

$$(\neg A \vee \neg B) \wedge (B \vee \neg A)$$

- How many vertices?
- How many directed edges?
- Is there any directed path from X to $\neg X$, for some literal X?
- Is there any directed cycle containing both X and $\neg X$, for some literal X?
- Is this 2CNF expression satisfiable?

Note - you only need to construct the graph for the examlet today!

An easy induction proof

Theorem: For all $n \in N$, $\sum_{i=1}^{n} i = n(n+1)/2$

Proof. We prove this theorem by induction.
Base case: The base case is $n = 1$. The left hand side has value 1, and so does the right hand side.
Our **inductive hypothesis** is that for some positive integer K, and for all positive integers k with $1 \leq k \leq K$, $\sum_{i=1}^{k} i = k(k+1)/2$.

An equivalent way of stating the inductive hypothesis is:
$\exists K \in Z^{+}$ such that for all $k \in \{1, 2, \ldots, K\}$, $\sum_{i=1}^{k} i = k(k+1)/2$.
We now wish to show that this statement is also true for $K + 1$; i.e.,

$$\sum_{i=1}^{K+1} i = (K+1)(K+2)/2.$$

Easy induction proof, continued

By definition $\sum_{i=1}^{K+1} i = \sum_{i=1}^{K} i + (K+1)$
By the inductive hypothesis,

$$\sum_{i=1}^{K} i = K(K+1)/2$$

Hence,

$$\sum_{i=1}^{K+1} i = \sum_{i=1}^{K} i + (K+1)$$

$$= K(K+1)/2 + (K+1) = (K+1)(K+2)/2$$

Since K was arbitrary, our theorem is proved.
Q.E.D.

Necessary parts of induction proofs

You want to prove $P(n)$ true for all $n \in \{n_0, n_0 + 1, \ldots\}$ (i.e., for all $n \in Z^+, n \geq n_0$), where $P(n)$ is some property that is either true or false.

- Inductive Hypothesis: Let n_1 be the largest base case we look at. Then the inductive hypothesis is: $\exists K \in Z, K \geq n_1$ such that for all $n \in \{n_0, n_0 + 1, \ldots K\}$, $P(n)$ is true.
- Base case (sometimes more than one): show $P(n)$ is true for $n_0, n_0 + 1, \ldots, n_1$. Note: it might be enough to just show it for n_0, but this depends on the problem.
- Use the fact that $P(n)$ is true for all n between n_0 and K to prove $P(K + 1)$ is true.
- Point out that K was arbitrary so the result holds for all K.
- Optional: say "Q.E.D."

The Inductive Hypothesis

The inductive hypothesis must be a statement that that depends on a parameter K and that is either true or false.

Your inductive hypothesis must be that the statement is true for all values n between some base case n_0 and some specific (but arbitrary) value K.

A good inductive hypothesis: $\exists K \in Z^+, K \geq n_1$ such that $\forall n \in \{n_0, n_0 + 1, \ldots, K\}$, property $P(n)$ is true.

Note: we always begin the IH with $\exists K$.

Using induction for recursively defined functions

Simplest examples of induction proofs are for recursively defined functions or sets.

Let F be a function from the set $\{5, 6, \dots, \}$ to the real numbers, defined by

- $F(5) = 3$
- $F(n) = F(n-1) + 4.1$ if $n > 5$

We want to find a closed form solution for $F(n)$. A little work shows $F(n) = 3 + 4.1(n-5)$. To prove this by induction we think about:

- The statement we want to prove is for all integers $n \geq 5$ (not for all positive integers). Hence our base case will be just this value, $n = 5$.
- $F(n)$ depends only on $F(n-1)$. Hence we could use "weak induction". But we will use strong induction anyway, just to practice.

Induction proof

- $F(5) = 3$
- $F(n) = F(n-1) + 4.1$ if $n > 5$

The inductive hypothesis is

$\exists K \in Z, K \geq 5, \text{ such that } \forall n \in \{5, 6, \ldots, K\}\ F(n) = 3 + 4.1(n-5)$

Note the form of the IH depends on the base case (n_0) and has the following form:

$$\exists K \in Z, K \geq n_0, \text{ such that } \forall n \in \{n_0, n_0 + 1, \ldots, K\}\ P(n)$$

where $P(n)$ is something that is true or false for a given n.

Examlet

Note:

- Closed book: no notes, no extra paper, no calculators, no laptops, no cellphones. Put all this away, and under your seat if possible.
- This exam is under the honor system.
- Graded examlets will be available starting Monday morning in Siebel 0211. Please attend office hours in 0211 to get your examlet back.