CS173 Lecture B, September 1, 2015

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Logic exercises

- Simplifying logical expressions, and seeing when two logical expressions are equivalent
- Determining if a logical expression can be *satisfied*
- Expressing English statements in logic

Simplifying logical expression

Objectives:

- Remove all unnecessary parentheses
- Remove all $\rightarrow$ or $\leftrightarrow$

Hence, you need to be able to simplify expressions like

$$\neg(a \rightarrow b) \vee (\neg b)$$

Simplifications (warm-up)

When A and B are logical expressions, and you say $A \equiv B$, you mean that they have the same truth values. (You can also write this as $A \leftrightarrow B$.)

For example:

- $\neg\neg x \equiv x$ (obvious)
- $x \vee (x \wedge y) \equiv x$

  Similarly, you can write $x \vee (x \wedge y) \leftrightarrow x$. In other words, $x \vee (x \wedge y)$ is true if and only if x is true.
- $x \vee \neg x \equiv T$

  In other words, $x \vee \neg x$ is always true, no matter what x is.
- $x \wedge \neg x \equiv F$

  In other words, $x \wedge \neg x$ is never true, no matter what x is.

Truth Tables

We (sometimes) use truth tables to check our analyses. Here's an example of a very simple truth table for the expression $A \wedge B$:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A more complicated truth table

Consider the expression $[(A \rightarrow B) \wedge \neg B] \rightarrow A$. Is this always true? Sometimes true and sometimes false? Always false? Let's use a truth table to answer this.

A	B	$(A \rightarrow B) \wedge \neg B$	$[(A \rightarrow B) \wedge \neg B] \rightarrow A$
T	T	F	T
T	F	F	T
F	T	F	T
F	F	T	F

So the answer is that it is sometimes true and sometimes false. Note that we also showed $[(A \rightarrow B) \wedge \neg B] \rightarrow A \equiv A \vee B$.

De Morgan's Laws

- Negation of $A \wedge B$: $\neg A \vee \neg B$
  This is also written as
  $$\neg(A \wedge B) \equiv \neg A \vee \neg B$$
  or as
  $$\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$$
- Negation of $A \vee B$: $\neg A \wedge \neg B$
  This is also written as
  $$\neg(A \vee B) \equiv \neg A \wedge \neg B$$
  or as
  $$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$$

Negation, warm up with quantifiers

- Negation of $\forall x \in S, P(x)$:

  $\exists x \in S$ s.t. $\neg P(x)$

  Negation of $\exists x \in S$ s.t. $P(x)$

  - $\forall x \in S, \neg P(x)$

Negating Logical Expressions

Consider the expression

$$A \rightarrow B$$

To negate this, we have:

$$\neg(A \rightarrow B)$$

$$\equiv \neg(\neg A \vee B)$$

$$\equiv \neg\neg A \wedge \neg B$$

$$\equiv A \wedge \neg B$$

Negation, again

Negate: $(x \rightarrow y) \wedge \neg x$
First Solution:

$$\neg[(x \rightarrow y) \wedge \neg x]$$

$$\equiv \neg(x \rightarrow y) \vee \neg\neg x$$

$$\equiv \neg(\neg x \vee y) \vee x$$

$$\equiv (\neg\neg x \wedge \neg y) \vee x$$

$$\equiv (x \wedge \neg y) \vee x$$

$$\equiv x$$

Negation, again

Negate: $(x \rightarrow y) \wedge \neg x$

Second Solution: We begin by simplifying the expression above before negating it. Note that

$$x \rightarrow y \equiv \neg x \vee y$$

Hence

$$(x \rightarrow y) \wedge \neg x$$

$$\equiv (\neg x \vee y) \wedge \neg x$$

$$\equiv (\neg x \wedge \neg x) \vee (y \wedge \neg x)$$

$$\equiv \neg x \vee (y \wedge \neg x)$$

$$\equiv \neg x$$

Therefore,

$$\neg[(x \rightarrow y) \wedge \neg x] \equiv \neg\neg x \equiv x$$

Satisfiability

Some logical expressions can never be true, some are always true, and some depend on the values of their variables. **T** and **F** refer to the logical constants True and False, respectively. Examples:

1. $A \vee \neg A$ (always true)
2. $A \wedge \neg A$ (never true)
3. $A \vee B$ (sometimes true and sometimes false, depends on A and B)
4. $A \wedge F$ (never true)

Statements that are always true are called *tautologies*. Statements that can be true (or are always true) are said to be *satisfiable*, and otherwise they are said to be *unsatisfiable*.

Satisfiability

For each of the following expressions, determine if it is satisfiable or not satisfiable. If it is satisfiable, determine if it is a tautology.

1. $(A \wedge B) \rightarrow A$
2. $(A \wedge B) \rightarrow \neg A$
3. $(A \wedge B) \leftrightarrow A$
4. $(A \rightarrow B) \wedge A \wedge \neg B$
5. $A \rightarrow \neg A$

Satisfiability

For each of the following expressions, determine if it is satisfiable or not satisfiable. If it is satisfiable, determine if it is a tautology.

1. $(A \wedge B) \rightarrow A$
   (Answer: tautology)
2. $(A \wedge B) \rightarrow \neg A$
   (Answer: satisfiable (A = B = **F**) but not a tautology (A = B = **T**))
3. $(A \wedge B) \leftrightarrow A$
   (Answer: satisfiable (A = B = **T**) but not a tautology (A = **T** and B = **F**)
4. $(A \rightarrow B) \wedge A \wedge \neg B$
   (Answer: not satisfiable, so never true)
5. $A \rightarrow \neg A$
   (Answer: satisfiable (A = **F**) but not a tautology (A = **T**))

Conjunctive Normal Form (CNF)

A logical expression of the form

$$A_1 \vee A_2 \vee A_3 \vee ... \vee A_k$$

where the A_i are literals (statement letters or their negations) is called a *disjunctive clause*.
Then

$$C_1 \wedge C_2 \wedge C_3 \wedge \ldots \wedge C_p,$$

where each C_i is a disjunctive clause, is said to be in *conjunctive normal form*, or CNF.
CNF is very popular in computer science!

Two-satisfiability

A special case of CNF is where each clause has at most two literals! That is, expression that are written in the form $(A_1 \vee B_1) \wedge (A_2 \vee B_2) \wedge \ldots (A_k \vee B_k)$.
Which of the following CNF expressions are satisfiable?

1. $(x \vee y) \wedge (\neg x \vee \neg y)$
2. $(x \vee y) \wedge (\neg x \vee \neg y) \wedge x$
3. $(x \vee y) \wedge (\neg x \vee \neg y) \wedge x \wedge y$
4. $(x \vee y) \wedge (\neg x \vee \neg z) \wedge (\neg y \vee z) \wedge (\neg x \vee z)$
5. $(\neg x \vee y) \wedge (\neg y \vee z) \wedge (\neg z \vee x) \wedge (x \vee z)$