CS173 Lecture B, October 1, 2015

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Preparation

Let $A = \{a_1, a_2, \ldots, a_n\}$. Then

- $\sum_{a \in A} a = a_1 + a_2 + \ldots + a_n$
- $\prod_{a \in A} a = a_1 \times a_2 \ldots \times a_n$
- $\max\{a \in A\} = \max(A)$ is the element a_j of A such that $a_j \geq a_i$ for all $i, 1 \leq i \leq n$
- $\min\{a \in A\} = \min(A) = a_j$ such that $a_j \leq a_i$ for all $i, 1 \leq i \leq n$

Example: $A = \{1, 3, 7, 8\}$

- $\sum_{a \in A} a = 19$
- $\prod_{a \in A} a = 168$
- $\max(A) = 8$
- $\min(A) = 1$

Sample problem

Let $M : N \times N \rightarrow Z$ be defined by

- $M(0, j) = 2j$
- $M(i, 0) = i$
- $M(i, j) = \min\{M(i-1, j) + 1, M(i, j-1) + 2\}$

Do the following:

- Compute $M(i, j)$ for $0 \leq i, j \leq 3$
- Prove (using induction on $i + j$) that for all natural numbers i, j, $M(i, j) = i + 2j$.

The proof by induction

The base case is $i + j = 0$, so $i = j = 0$.
By definition, if $i = j = 0$, then $M(i, j) = 0 = i + 2 \times j$. Hence the assertion is true for $i = j = 0$.
The inductive hypothesis is $\exists K \in N$ such that for all natural numbers i, j such that $i + j \leq K$, $M(i, j) = i + 2j$.
Note that $K \geq 0$, since we proved the base case $(K = 0)$.

Induction proof, continued

We want to show that if $i + j = K + 1$, then $M(i, j) = i + 2j$.
Note:
If $i = 0$, then $M(i, j) = 2j = i + 2j$ and so the assertion holds.
If $j = 0$ then $M(i, j) = i = i + 2j$ and so the assertion holds.
Now assume i, j are both at least 1. Hence, by definition,
$M(i, j) = \min\{M(i, j-1) + 2, M(i-1, j) + 1\}$.
Then since $i + j - 1 = K < K + 1$, we can apply the inductive hypothesis to $M(i, j-1)$ and $M(i-1, j)$.
Hence,

$$M(i, j-1) + 2 = i + 2(j-1) + 2 = i + 2j$$
$$M(i-1, j) + 1 = i - 1 + 2j + 1 = i + 2j$$

Therefore, $M(i, j) = i + 2j$, which is what we wanted to show.
Since K was arbitrary, the result holds for all natural numbers i, j.