These notes show two ways of proving theorems - one by induction and the other by contradiction.

**Theorem 1:** Let \( F : \mathbb{Z}^+ \to \mathbb{Z} \) be defined recursively by

- \( F(1) = 3 \)
- \( F(n) = 2F(n - 1) + 1 \) for \( n \geq 2 \)

Then \( F(n) > 2^{n-1} \) for all \( n \in \mathbb{Z}^+ \).

**Proof 1:** The proof is by contradiction. Let \( P(n) \) be the assertion that \( F(n) > 2^{n-1} \). Suppose it is not the case that \( P(n) \) is true for all positive integers \( n \). Then there is at least one positive integer where \( P(n) \) is false. Let \( N \) be the first such positive integer. Hence \( F(N) \leq 2^{N-1} \) and \( N \geq 1 \).

We first check if \( N = 1 \) is possible. By the definition of the function, \( F(1) = 3 \) and \( 3 > 2^{1-1} = 2^0 = 1 \). Hence \( P(1) \) is true, and so \( N = 1 \) is not possible. Therefore \( N \) must be at least 2.

Since \( N \geq 2 \), by the definition of the function, we see that

\[
F(N) = 2F(N - 1) + 1.
\]

Note that \( N - 1 \geq 1 \) (because \( N \geq 2 \)) and so \( P(N - 1) \) is true (because \( N \) is the smallest positive integer \( n \) for which \( P(n) \) is false). Therefore

\[
F(N - 1) > 2^{N-2}.
\]

Putting these together, we obtain

\[
F(N) = 2F(N - 1) + 1 > 2 \times 2^{N-1} + 1 = 2^N + 1 > 2^N.
\]

In other words, we have shown that \( P(N) \) is also true. Thus, we derived a contradiction, and so the statement \( P(n) \) must be true for all positive integers \( n \). Q.E.D.

**Proof 2:** The second proof is by induction on \( n \). Let \( P(n) \) be as in the previous proof, and note that we have already established that \( P(1) \) is true.

Let \( N \) be an arbitrary positive integer. Our Inductive hypothesis is that \( P(N) \) is true, and we wish to derive that \( P(N + 1) \) is true. In other words, we wish to derive that \( F(N + 1) > 2^N \).

Since \( N \geq 1 \), it follows that \( N + 1 \geq 2 \), and hence by the definition of the function \( F \), we obtain:

\[
F(N + 1) = 2F(N) + 1
\]

By our I.H., \( F(N) > 2^{N-1} \), and so

\[
F(N + 1) = 2F(N) + 1 > 2 \times 2^{N-1} + 1 = 2^N + 1 > 2^N.
\]
In other words, we have shown that

\[ F(N + 1) > 2^N \]

and thus \( P(N + 1) \) is true.

Since \( N \) was an arbitrary positive integer, this means we have shown that \( P(n) \) is true for all positive integers \( n \). Q.E.D.