CS173 Lecture B

Tandy Warnow

December 1, 2015

CS 173, Lecture B

Review of Cardinality and Countability

Tandy Warnow

Today

- The final exam
- Cardinality of infinite sets (review)
- Countability and how to prove that a set is countable (review)
- Uncountability, and how to prove that a set is not countable (review)
- ICES forms (need a student volunteer to collect and return them)

What won't be on the final exam

What won't be on the exam:

- Dynamic programming
- The graph algorithm for 2-SAT
- Genome assembly and de Bruijn graphs

The final exam will contain problems on everything else.

- Some of the exam requires proofs (by induction, by contradiction, etc.).
- Some of it will be multiple choice or True/False.
- You will need to know definitions.
- Nearly all the problems will be verbatim (or almost verbatim) from the exam review document.

So just make sure you can do what's on the exam review document!

Examlet on Thursday, December 3

It will have three problems:

1. One proof that a set is countable or perhaps uncountable
2. One problem involving proving a theorem about trees
3. One problem where you are given an adjacency matrix and asked to draw the graph it defines, and then find various things. Examples of those things might include:
   - Breadth First Search tree starting at a particular node
   - minimum dominating set
   - minimum vertex cover
   - maximum matching
   - maximum independent set
   - maximum clique
   - minimum vertex coloring,
   - Hamiltonian path (if it exists)
   - Eulerian circuit (if it exists)

Material about Countability/Uncountability

Definitions:

- Countable and uncountable
- Finite and infinite

Techniques:

- Cantor's diagonalization (proves uncountability)
- Zig-zag (proves countability)

Theorems:

- The countable union of countable sets is countable (zig-zag technique).
- The finite product of countable sets is countable (zig-zag technique for two sets, induction to generalize).
- Any subset of a countable set is countable.
- Any superset of an uncountable set is uncountable.

Cardinality

The **cardinality** of a finite set X is the number of elements in X, and is denoted $|X|$.
Hence, $|\{1, 2, 3, 4, 5\}| = |\{2, 9, 12, 17, 18\}|$.
A set X is **finite** if $|X| = n$ for some $n \in Z$.
A set X is **infinite** if there does not exist any $n \in Z$ so that $|X| = n$.

Note, an alternative definition for infinite is that $\exists Y \subset X$ and a bijection $f : X \to Y$.
How do we talk about cardinality of infinite sets?
When can we say that $|X| = |Y|$ for infinite sets $|X|$ and $|Y|$?

Cardinality of infinite sets

We will say that $|X| = |Y|$ if $\exists f : X \to Y$ where f is a bijection.

A set X is **countably infinite** if there is a bijection from X to $\mathbb{N}$.

Using the prior notation, we say X is countably infinite if $|X| = |\mathbb{N}|$.

In the textbook, the definition of countable is given as "finite or countably infinite."

But some authors use countable *only* for countably infinite... be aware of this.

Proving a set is countable

There are several techniques to prove a set S is countable.

1. Provide a bijection between S and $\mathbb{N}$
2. Provide injections (1-1 functions) in each direction between S and $\mathbb{N}$

Or, equally validly, describe an algorithm that produces an enumeration of the elements of S.

Proving that $\mathbb{Z}^-$ is countable

To prove that $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$ is countable, we define a bijection $f : \mathbb{Z}^- \to \mathbb{N}$.

- Let $f(n) = |n| - 1$. We prove that f is a bijection between $\mathbb{Z}^-$

  and $\mathbb{N}$.
  - (f ix $1-1$:) Suppose $f(x) = f(y)$. Then $|x| - 1 = |y| - 1$, and so $|x| = |y|$. Since $\{x, y\} \subset \mathbb{Z}^-$, both x and y are negative. Hence $x = y$, and so f is $1-1$.
  - (f is onto:) Let $n \in \mathbb{N}$; hence $n \geq 0$. Let $n' = -(n+1)$. Note that $f(n') = |-(n+1)| - 1 = n + 1 - 1 = n$. Hence f is onto.

Hence $\mathbb{Z}^-$ is countable.

Proving that $\mathbb{Z}$ is countable

Here we show that $\mathbb{Z}$ is countable. We have three options:

1. Provide a bijection between $\mathbb{Z}$ and $\mathbb{N}$
2. Provide injections (1-1 functions) in each direction between $\mathbb{Z}$ and $\mathbb{N}$
3. Or, equally validly, describe an algorithm that produces an enumeration of the elements of $\mathbb{Z}$.

Let's do the third one.

We will alternate between listing the non-negative integers ($\mathbb{N}$) and the negative integers ($\mathbb{Z}^-$).

So:

$$0, -1, 1, -2, 2, -3, 3, \ldots$$

This is a perfectly valid proof that $\mathbb{Z}$ is countable!

The Zig-Zag Technique

Suppose A and B are both countable. How do we prove that $A \cup B$ is countable?

Zig-zag technique:

- Step 1: List the elements of A and B.
- Step 2: List the elements of $A \cup B$ by alternating between the elements of A and B.

Because A is countable, we can list its elements, $A = \{a_0, a_1, a_2, \ldots, \}$. Ditto for B, so $B = \{b_0, b_1, b_2, \ldots, \}$.

Enumerate $A \cup B$ by alternating between elements of A and B: $A \cup B = \{a_0, b_0, a_1, b_1, a_2, b_2, \ldots\}$.

Finite unions of countable sets are countable

Suppose $A_1, A_2, \ldots, A_k$ are all countable. How do we prove that $\cup_{i=1}^{k} A_i$ is countable?

Enumerate the elements of the finite union as follows:

- List the first element from each of the k sets
- Then list the second element from each of the k sets
- Then list the third element from each of the k sets
- and so on

Let $A_i = \{a_{i0}, a_{i1}, a_{i2}, \ldots,\}$, $i = 1, 2, \ldots, k$.

We obtain the list

$$\cup_{i=1}^{k} A_i = \{a_{10}, a_{20}, \ldots, a_{k0}, a_{11}, a_{21}, \ldots, a_{k1}, \ldots\}$$

Countable unions of countable sets are countable

We will use the zig-zag technique to prove that $\cup_{i \in \mathbb{N}} A_i$ is countable, if each A_i is countable.

Consider the infinite by infinite matrix M, where A_0 is in the first row, A_1 is in the second row, etc.

The i^{th} row lists the elements of A_i.

We need to enumerate all the elements of $\cup_i A_i$.

Countable unions of countable sets are countable

Each A_i is countable, $i \in \mathbb{N}$. We list all the elements of A_i as the i^{th} row in an infinite by infinite matrix, M.

We then use the **zig-zag** technique to enumerate all the elements, visiting them as follows:

- $M[0, 0]$
- $M[1, 0], M[0, 1]$
- $M[2, 0], M[1, 1], M[0, 2]$
- etc.

Note that because some elements may appear in more than one set, we may list an element more than once. We'll just skip the element if it's already appeared in our list.

Countable unions of finite sets are countable

Suppose A_i is a finite set for $i = 0, 1, 2, \ldots$

To prove that $\cup_i A_i$ is countable, we can just list each set in the order we see it.
Hence, we begin with A_0, then we list A_1, then we list A_2, etc.

The product of two countable sets is countable

Suppose A and B are countable. Is $A \times B$ countable?

Use the zig-zag technique.

Let $A = \{a_0, a_1, \ldots,\}$ and $B = \{b_0, b_1, \ldots,\}$.

Define matrix $M[i,j] = (a_i, b_j)$.

Enumerate the elements of $A \times B$ using the zig-zag technique.

The product of finitely many countable sets is countable

Suppose A_i is countable for each $i = 0, 1, 2, \ldots, k$. We prove $\prod_{i=0}^{k} A_i$ countable by induction on k.

- Base case: $k = 2$ (see previous slide)
- Inductive hypothesis: $\exists K \in Z^+$ such that $\forall k, 1 \leq k \leq K$, $\prod_{i=1}^{k} A_i$ is countable if each A_i is countable.
- By definition, $\prod_{i=0}^{K+1} A_i = \prod_{i=0}^{K} A_i \times A_{K+1}$.
- By the inductive hypothesis, $\prod_{i=0}^{K} A_i$ is countable.
- By the inductive hypothesis, the product of two countable sets is countable.
- Hence $\prod_{i=0}^{K+1} A_i = \prod_{i=0}^{K} A_i \times A_{K+1}$ is countable.

So far

We have shown we can prove countability of a set S

- by demonstrating a bijection between S and $\mathbb{N}$
- by demonstrating 1-1 functions between S and $\mathbb{N}$ in both directions
- by defining an infinite two-dimensional matrix containing all the elements of the set, and using a zig-zag technique to list the elements

Hence:

- The countable union of countable sets is countable
- The finite product of countable sets is countable
- If $X \subseteq S$ and S is countable, then so is X
- If $|X| \leq |S|$ and S is countable, then so is X

What about uncountability?

A set X is uncountable if X is infinite but $|X| \neq |\mathbb{N}|$.
Examples:

- $[0, 1]$
- $\mathbb{R}$
- $\mathbb{P}(\mathbb{N})$
- The set of functions from $\mathbb{N}$ to $\{0, 1\}$

Furthermore, for any set A that is listed above, then

- Any set X that contains A as a subset is uncountable
- Any set X that contains a subset Y where $|Y| = |A|$ is uncountable

Why $\mathbb{P}(\mathbb{N})$ is uncountable

The proof that $\mathbb{P}(\mathbb{N})$ is uncountable is in the book, and we've gone over it already.

Why $[0, 1]$ is uncountable

Proof by contradiction.

Suppose it is countable. Then we can list the elements as $a_0, a_1, a_2, \ldots$.

Write out the matrix, with a_i the i^{th} row, and give each number in decimal notation, written as 0.y (for some y). Note that this is possible, even for x=1! (In other words, $1.0 = 0.\bar{9}$)

Why $[0, 1]$ is uncountable

Hence, if $a_0 = 0.1701$, then the first row would look like:

$$1, 7, 0, 1, 0, 0, 0, \ldots$$

If $a_1 = \frac{1}{3} = 0.\overline{3}$ then the second row would be:

$$3, 3, 3, 3, 3, 3, 3, \ldots$$

if $a_2 = 1$, we list it as

$$9, 9, 9, 9, 9, 9, 9, \ldots$$

We have to define a real number z that is not in this matrix. How do we do this?

Why $[0, 1]$ is not countable

We set the digits for z by setting z_i so that it is different from every element in the matrix in the i^{th} position.

We set be $z_i = M[i, i] + 2(mod 10)$.

Why $[0, 1]$ is not countable

The mapping we used is:

- If $M[i, i] = 0$ we set $z_i = 2$
- If $M[i, i] = 1$ we set $z_i = 3$
- If $M[i, i] = 2$ we set $z_i = 4$
- If $M[i, i] = 3$ we set $z_i = 5$
- If $M[i, i] = 4$ we set $z_i = 6$
- If $M[i, i] = 5$ we set $z_i = 7$
- If $M[i, i] = 6$ we set $z_i = 8$
- If $M[i, i] = 7$ we set $z_i = 9$
- If $M[i, i] = 8$ we set $z_i = 0$
- If $M[i, i] = 9$ we set $z_i = 1$

Finishing the proof

Now we derive the contradiction!

- We assumed that the set $[0, 1]$ is countable, and that matrix M has a row for every element in the set.
- We defined the number z based on its decimal expansion, so that $z_i \neq M[i, i]$ for any i.
- Therefore the matrix M cannot have a row for every element of $[0, 1]$.
- Hence we derive a contradiction, and $[0, 1]$ must be uncountable.

Why didn't we use $z_i = M[i,i] + 1 (mod 10)$?

Note how we defined z... our mapping...
Suppose we had set $z_i = M[i,i] + 1 (mod 10)$.

Note that $.1 = .0\bar{9}$ (they are really the same number).

Suppose we had obtained $z = .1$ because the diagonal elements were $0, 9, 9, 9, 9, \ldots$.

The first row could have been for the element $0.0\bar{9}$.

Proving a set is uncountable

To prove a set X is uncountable, do one of the following:

- The same kind of proof by contradiction – enumeration and diagonalization
- Prove that $|X| = |Y|$ where Y is uncountable
- Find an uncountable set Y and show that $Y \subset X$
- Find an uncountable set Y and a 1-1 function from Y to X; this is denoted by $|Y| \leq |X|$

Other uncountable sets

Using the diagonalization argument, or using other theorems, or combining these techniques with proofs by contradiction, we can now prove that the following sets are uncountable:

- $\mathbb{P}(Y)$, where Y is any infinite set
- The set of functions from A to X, where A is countable and $|X| \geq 2$ (e.g., $A = \mathbb{Z}$ and $X = \{1, 2, 3\}$).
- $\mathbb{R}$
- $\mathbb{R} \setminus \mathbb{Q}$

Summary

Definitions:

- Countable and uncountable
- Finite and infinite

Techniques:

- Cantor's diagonalization (proves uncountability)
- Zig-zag (proves countability)

Theorems:

- The countable union of countable sets is countable (zig-zag technique).
- The finite product of countable sets is countable (zig-zag technique for two sets, induction to generalize).
- Any subset of a countable set is countable.
- Any superset of an uncountable set is uncountable.