CS173 Lecture B

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November 22, 2015

CS 173, Lecture B

Cardinality and Countability

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Conflict Exam?

If you have another final exam at the same time (December 11, 8-11 AM), you need to send me (by email) the full information about the course, and also send me *all the times you have final exams.*

Margaret and I will determine whether I have the responsibility to give you an alternate time and place for the exam, or whether the other instructor has that responsibility.

Send this email by tomorrow (Friday, November 20), so we can figure out how to deal with this.

Final Exam stuff

- The final exam in two rooms: 1320 DCL and 1KAM (room 62, Krannert Art Museum).
  - Discussion Sections BDA, BDC, and BDF go to DCL.
  - Discussion Sections BDB, BDD, and BDE go to Krannert Art Museum.

Final Exam material

- Most of the exam will be identical or nearly identical to problems in the homeworks, reading quizzes, and examlets, or problems that we solved in the lecture.
- The best thing you can do is to come to office hours to understand why you got points off your examlets, reading quizzes, and homeworks; this is particularly important for examlet questions where you were asked to do a proof. If you missed points on a proof and you aren't sure what you did wrong, come to office hours.
- Come to 173 Lecture B staff for problems related to (a) genome assembly and de Bruijn graphs, (b) the graph algorithm for 2SAT, (c) dynamic programming. Everything else should be okay for any 173 staff person (and probably even the dynamic programming problems).

Today

- Cardinality of infinite sets
- Countability and how to prove that a set is countable
- Uncountability, and how to prove that a set is not countable

Cardinality

The **cardinality** of a finite set X is the number of elements in X, and is denoted $|X|$.
Hence, $|\{1, 2, 3, 4, 5\}| = |\{2, 9, 12, 17, 18\}|$.
A set X is **finite** if $|X| = n$ for some $n \in Z$.
A set X is **infinite** if there does not exist any $n \in Z$ so that $|X| = n$.

Note, an alternative definition for infinite is that $\exists Y \subset X$ and a bijection $f : X \rightarrow Y$.
How do we talk about cardinality of infinite sets?
When can we say that $|X| = |Y|$ for infinite sets $|X|$ and $|Y|$?

Cardinality of infinite sets

We will say that $|X| = |Y|$ if $\exists f : X \to Y$ where f is a bijection.

A set X is **countably infinite** if there is a bijection from X to $\mathbb{N}$. Using the prior notation, we say X is countably infinite if $|X| = |\mathbb{N}|$.

Note that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Z}^+|$.

Proof

We prove that $|\mathbb{Z}| = |\mathbb{N}|$ by establishing a bijection from $\mathbb{Z}$ to $\mathbb{N}$. We will send the non-negative integers to the even natural numbers, and the negative integers to the odd natural numbers.

- $f(x) = 2x$ when $x \geq 0$
- $f(x) = 2|x| - 1$ when $x < 0$

Similarly, you can come up with bijections between every other pair of the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Z}^+$, to prove that they all have the same cardinality.

When is $A \times B$ countable?

Suppose A and B are both countable sets. Is $A \times B$ countable?

Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be the enumeration of these sets.

Can we enumerate this set so that every element appears in some finite index?

When is $A \times B$ countable?

Consider the infinite matrix $M[i,j]$ where $M[i,j]$ corresponds to the ordered pair (a_i, b_j).

Consider the enumeration of the set $A \times B$, given by going down short diagonals (right to left, decreasing):

- $M[1,1]$
- $M[1,2], M[2,1]$
- $M[1,3], M[2,2], M[3,1]$
- $M[1,4], M[2,3], M[3,2], M[4,1]$
- etc.

Note that every element of $A \times B$ appears at some finite index, and so enumeration defines a bijection between the elements of $A \times B$ and Z^+.

Hence if A and B are countable, then $A \times B$ is countable.

General properties

- If $|X| \leq |Y|$ and Y is countable, then X is countable.
- If $X_1, X_2, \ldots, X_k$ are each countable, then $\prod_i X_i$ is countable.
- If $X_1, X_2, \ldots, X_k$ are each countable, then $\cup_i X_i$ is countable.

Hence $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$ are both countable.

Proving $|A| = |B|$

The **Cantor-Schroeder-Bernstein Theorem** theorem shows that for any two sets A, B, $|A| = |B|$ whenever you can find two $1-1$ functions, one from A to B, and the other from B to A.

More specifically, they show that if you have two $1-1$ functions, then there is a *bijection* between the two sets.

Finding two $1-1$ functions is generally easier to do than finding a bijection.

Using Cantor-Schroeder-Bernstein Theorem

For example, to prove $|\mathbb{N}| = |\mathbb{Z}|$, we can write

- $f : \mathbb{N} \to \mathbb{Z}$, where $f(x) = x$
- $g : \mathbb{Z} \to \mathbb{N}$, where
  - $g(x) = 2x$ if $x \geq 0$
  - $g(x) = 2|x| + 1$ if $x < 0$

It's easy to see that f and g are both $1-1$, so by the Cantor-Schroeder-Bernstein theorem, $|\mathbb{N}| = |\mathbb{Z}|$.

Cardinality of infinite sets

Consider the binary relation on sets $(X, Y) \in R$ if and only if $|X| = |Y|$.
Note that $|A| = |B|$ and $|B| = |C|$ implies that $|A| = |C|$.
It is easy to see that R is an equivalence relation!

Other sets

Are any of these sets countable?

- The set of functions from A to X, where A is countable and X is finite (e.g., $A = \mathbb{Z}$ and $X = \{1, 2, 3\}$).
- $\mathbb{P}(Y)$, where Y is a finite set
- $\mathbb{P}(Y)$, where Y is a countable set
- The interval $[0, 1]$
- $\mathbb{R}$
- $\mathbb{R} \setminus \mathbb{Q}$

Uncountable sets

A set X is uncountable if X is infinite but $|X| \neq |\mathbb{N}|$.
Examples:

- $[0, 1]$
- $\mathbb{R}$
- $\mathbb{P}(\mathbb{N})$
- The set of functions from $\mathbb{N}$ to $\{0, 1\}$

Furthermore, for any set A that is listed above, then

- Any set X that contains A as a subset is uncountable
- Any set X that contains a subset Y where $|Y| = |A|$ is uncountable

Why $\mathbb{P}(\mathbb{N})$ is uncountable

The proof that $\mathbb{P}(\mathbb{N})$ is uncountable is in the book, but we'll go over it here.

Why $\mathbb{P}(\mathbb{N})$ is uncountable

Proof by contradiction.
If $\mathbb{P}(\mathbb{N})$ is countable, then there is a bijection between $\mathbb{P}(\mathbb{N})$ and $\mathbb{N}$, and so we can list these sets $A_0, A_1, A_2, \ldots$.
We will write down these sets in a matrix format with entries 0 and 1, where A_i is represented by i^{th} row.
Hence, $M[i, j] = 1$ if and only if $j \in A_i$.

The matrix M

Recall that $M[i,j] = 1$ if and only if $j \in A_i$.
Example: let's suppose that the first four sets are $A_0 = \{0,3,5\}$, $A_1 = \{2,3\}$, $A_2 = \emptyset$, $A_3 = \{x \in \mathbb{N} : x \geq 3\}$

What do the first four rows of the matrix M look like?

The matrix M

Recall that $M[i,j] = 1$ if and only if $j \in A_i$.

Example: let's suppose that the first four sets are $A_0 = \{0, 3, 5\}$, $A_1 = \{2, 3\}$, $A_2 = \emptyset$, $A_3 = \{x \in \mathbb{N} : x \geq 3\}$

Let's construct $Y \subseteq \{0, 1, 2, 3\}$ so that $i \in Y$ if and only if $i \notin A_i$ for $i = 0, 1, 2, 3$. What is Y?

Diagonalization argument

We prove $\mathbb{P}(\mathbb{N})$ is uncountable using a diagonalization argument.

Consider the infinite matrix representing $\mathbb{P}(\mathbb{N})$.

By construction, every subset of $\mathbb{N}$ is represented by some row in the matrix.

Consider the set Y defined by $j \in Y$ if and only if $M_{j,j} = 0$.

Note that Y is a subset of $\mathbb{N}$.

Finishing the proof

Now we derive the contradiction!

- We assumed that the set $\mathbb{P}(\mathbb{N})$ is countable, and that matrix M has a row for every element in the set.
- We defined the set $Y \in \mathbb{P}(\mathbb{N})$ by $j \in Y$ if and only if $j \notin A_j$ for all $j \in \mathbb{N}$.
- Hence for all $j \in \mathbb{N}$, $Y \neq A_j$.
- Therefore the matrix M cannot have a row for every element of $\mathbb{P}(\mathbb{N})$.
- Hence we derive a contradiction.

Proving a set is uncountable

To prove a set X is uncountable, do one of the following:

- The same kind of proof by contradiction – enumeration and diagonalization
- Prove that $|X| = |Y|$ where Y is uncountable
- Find an uncountable set Y and show that $Y \subset X$
- Find an uncountable set Y and a 1-1 function from Y to X; this is denoted by $|Y| \leq |X|$

Other uncountable sets

Using the diagonalization argument, or using other theorems, we can now prove that the following sets are uncountable:

- $\mathbb{P}(Y)$, where Y is a countable set
- The set of functions from A to X, where A is countable and $|X| \geq 2$
- The interval $[0, 1]$
- $\mathbb{R}$
- $\mathbb{R} \setminus \mathbb{Q}$

Summary

What we covered today:

- Definition of cardinality for infinite sets
- Cantor-Schroeder-Bernstein Theorem
- Definition of countability
- How to prove countability
- Definition of uncountability
- Diagonalization proofs for uncountability
- Other techniques for proving uncountability