

CS 173
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Topics for today

- Basics of combinatorial counting
- Applications to running time analysis

Using combinatorial counting

- Evaluating exhaustive search strategies:
 - Finding maximum clique
 - Determining if a graph has a 3-coloring
 - Finding a maximum matching in a graph
 - Determining if a graph has a Hamiltonian cycle or an Eulerian graph

Combinatorial counting

How many ways can you

- put n items in a row?
- pick k items out of n ?
- pick subsets of a set of size n ?
- assign k colors to the vertices of a graph?
- match up n boys and n girls?

Technique

- To count the number of objects, design an algorithm to generate the entire set of objects. Check if each object is created exactly once (if not, you will have to do a correction later).
- The algorithm's output can be seen as the leaves of a decision tree, and you can just count the leaves.

Putting n items in a row

Algorithm for generating all the possibilities:

- For $i=1$ up to n , DO
 - Pick an item from S to go in position i
 - Delete that item from the set S

Analysis: each way of completing this generates a different list.

The number of ways of performing this algorithm is $n!$

Number of subsets of a set of size n

Algorithm to generate the subsets of a set

$$S = \{s_1, s_2, s_3, \dots, s_n\}$$

- For $i=1$ up to n DO:
 - Decide if you will include s_i

Analysis: each subset is generated exactly once, and the number of ways to apply the algorithm is $2 \times 2 \times \dots \times 2 = 2^n$.

k-coloring a graph

Let G have vertices $v_1, v_2, v_3, \dots, v_n$

Algorithm to k-color the vertices:

- For $i=1$ up to n DO:
 - Pick a color for vertex v_i

Analysis: each coloring is produced exactly once, and there are k^n ways of applying the algorithm.

Matching n boys and girls

Algorithm:

- Let the boys be B_1, B_2, \dots, B_n and let the girls be G_1, G_2, \dots, G_n .
- For $i=1$ up to n DO
 - Pick a girl for boy B_i , and remove her from the set

Analysis: there are n ways to pick the first girl, $n-1$ ways to pick the second girl, etc., and each way produces a different matching.

Total: $n!$

Picking k items out of n

Algorithm for generating all the possibilities:

- For $i=1$ up to k , DO
 - Pick an item from S to include in set A
 - Delete that item from the set S

The number of ways of performing this algorithm is $n(n-1)(n-2)\dots(n-k+1)=n!/(n-k)!$

But each set A can be generated in multiple ways - and we have overcounted!

Fixing the overcounting

Each set A of k elements is obtained through $k!$ ways of running the algorithm. As an example, we can generate $\{s_1, s_5, s_3\}$ in 6 ways, depending upon the order in which we pick each of the three elements.

So the number of different sets is the number of ways of running the algorithm, divided by $k!$.

The solution is $n!/[k!(n-k)!]$

Summary (so far)

- To count the number of objects, design an algorithm to generate the entire set of objects. Check if each object is created exactly once (if not, you will have to do a correction later).
- The algorithm's output can be seen as the leaves of a decision tree, and you can just count the leaves.

Summary

- Number of orderings of n elements is $n!$
- Number of subsets of n elements is 2^n
- Number of k -subsets of n elements is $n!/[k!(n-k)!]$
- Number of k -colorings of a graph is k^n

More advanced counting

- What is the number of k -subsets of a set $S = \{s_1, s_2, s_3, \dots, s_n\}$ that do not include s_1 ?
- What is the number of k -subsets of a set $S = \{s_1, s_2, s_3, \dots, s_n\}$ that do include s_1 ?
- What is the number of orderings of the set S in which s_1 and s_2 are not adjacent?
- What is the number of orderings of the set S in which s_1 and s_2 are adjacent?

New techniques

- Count the complement
- Divide into disjoint cases, and count each case

Example

- What is the number of k -subsets of a set $S = \{s_1, s_2, s_3, \dots, s_n\}$ that do not include s_1 ?
- Solution: same as number of k -subsets of $\{s_2, s_3, \dots, s_n\}$. So $(n-1)!/[(n-1-k)!k!]$

Example

- What is the number of k -subsets of a set $S = \{s_1, s_2, s_3, \dots, s_n\}$ that do include s_1 ?
- Solution: same as the number of $(k-1)$ -subsets of $\{s_2, s_3, \dots, s_n\}$, so
$$(n-1)! / [(n-k)!(k-1)!]$$

Example

- What is the number of orderings of the set S in which s_1 and s_2 are adjacent?
- Solution: two cases:
 - Case 1) s_1 followed by s_2
 - Case 2) s_2 followed by s_1
- Same number of each type. Easy to see that there are $(n-1)!$ of each type, so $2(n-1)!$ in total

Example

- Number of orderings of the set S in which s_1 and s_2 are not adjacent?
- This is the same as $n! - 2(n-1)!$

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