CS173, Trees

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October 23, 2018

CS 173

Trees

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Today's material

- Two theorems about trees with their proofs (comment about induction on trees)
- More theorems about trees (no proofs)

Theorems about trees

- Every tree $T = (V, E)$ with at least two vertices has at least two nodes that have degree 1 (hint: consider a longest path in the tree)
- If a tree $T = (V, E)$ has n vertices, then it has $n - 1$ edges

Every tree with at least two vertices has at least two leaves

The **leaves** of a tree are the nodes with degree 1; all other nodes are **internal nodes**.

Theorem: Every tree T with at least two vertices has at least two leaves.
Proof: Consider a longest path P in T.

Since T is finite, the path begins at some node v and ends at some node w.

We will prove that both endpoints of P are leaves.

Proving every tree with at least two vertices has at least two leaves

Proof by contradiction.
Let $T = (V, E)$ be a tree with at least two vertices, and let P be a longest path in tree T.
We write $P = v_1, v_2, \ldots, v_k$, with $k \geq 2$.

Suppose v_1 is not a leaf in T.
Then the neighbor set of v_1, denoted $\Gamma(v_1)$, has at least two vertices, and so $\exists x \in V, x \neq v_2$ such that $(v_1, x) \in E$.

- If x is in the path P, then $x = v_i$ for some $i > 2$, and $v_1, v_2, \ldots, v_i, v_1$ is a cycle in T, which is a contradiction (because T is a tree).
- If x is not in P, then $P' = x, v_1, v_2, \ldots, v_k$ is a path in T that is strictly longer than P, contradicting our hypothesis that P is a longest path.

Hence every endpoint of a longest path in T is a leaf, and so T contains at least two leaves.

Every tree with n vertices has exactly $n-1$ edges

Theorem: Every tree with n vertices has exactly $n-1$ edges.

Proof: By induction on n.
Base case: If $n = 1$, then T has no edges, and the base case holds.

The inductive hypothesis is that $\exists K \geq 1$ such that for all $n, 1 \leq n \leq K$, if tree T has n vertices then T has $n-1$ edges.

Now assume T has $K+1 \geq 2$ vertices; we want to prove T has K edges.

Proof that every tree with n vertices has $n-1$ edges

Since T is a tree, T has at least two leaves.

Let v be a leaf in T, and let w be its single neighbor.

Let T' be the graph created by deleting v.

Note that T' is a tree with K vertices, because:

- T' has one less vertex than T.
- T' is connected and acyclic

By the inductive hypothesis, T' has $K-1$ edges.

Recall that T' has one less edge than T.

Hence T has K edges. (q.e.d.)

Important: We started with a tree on $K+1$ vertices and removed a leaf to get a tree on K vertices. We did not go the reverse direction!

More about trees

What NP-hard problems can we solve efficiently on trees?

- Chromatic number?
- Max Clique?
- Maximum Independent Set?
- Minimum Dominating Set?
- Minimum Vertex Cover?

Some results on trees

- Chromatic number of any tree is at most 2.
- The max clique size of any tree is at most 2.
- For every tree T, there is at least one maximum independent set that contains all the leaves of T. (Why?)
- For every tree T on $n \geq 3$ leaves, there is a minimum vertex cover that does not contain any leaves. (Why?)
- For every tree T on $n \geq 3$ leaves, there is a minimum dominating set that does not contain any leaves. (Why?)

Class exercise

Do one or more of the following:

1. Prove every tree can be properly 2-colored.
2. Prove that every tree with at least 3 vertices has a minimum dominating set that does not contain any leaves.