CS 173, Relations

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Relations

Binary relations are sets of ordered pairs, expressing relationship (of some sort).

Thus, a binary relation R on a set S is a subset of $S \times S$.

Some examples of binary relations

Consider the set R of ordered pairs of integers where $(x, y) \in R$ if and only if x divides y.

Then $(3, 6) \in R$, and $(2, 2) \in R$, but $(6, 3) \notin R$. Thus, the order matters!

Questions:

- Is there an integer x such that $(x, y) \in R$ for all $y \in \mathbb{Z}$?
- Is there an integer p such that $(n, p) \notin R$ for all $n \in \mathbb{Z}$?
- Is there an integer p such that $(p, p) \notin R$?
- Is there an integer p such that $(p, 0) \in R$?
- Is there an integer p such that $(0, p) \in R$?

Class Exercise

Let $\mathbb{Z}$ denote the set of integers.

Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(x, y) \in R$ if and only if $x + y = 0$.

Questions:

- Give some examples of elements of R.
- Is it the case that $\forall x \in \mathbb{Z}$ and $\forall y \in \mathbb{Z}$, $(x, y) \in R \Rightarrow (y, x) \in R$?
- Is it the case that for all integers x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$?
- Is $(x, x) \in R$ for all $x \in \mathbb{Z}$?

More on binary relations

Properties of binary relations R on a set S that are often interesting.

- Reflexive: $\forall x \in S, (x, x) \in R$
- Irreflexive: $\forall x \in S, (x, x) \notin R$
- Symmetric: $\forall x \in S, \forall y \in S, (x, y) \in R \Rightarrow (y, x) \in R$
- Anti-symmetric: $\forall x \in S$ and and $\forall y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.
- Transitive: $\forall x \in S, y \in S, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$

Another relation

Let $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and let $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $R = \{(x, y) : x \in A, y \in A, x \leq y \leq x + 2\}$.

Questions:

- Give an example of some elements of R, and at least one ordered pair $(a, b) \notin R$ where $(a, b) \in A \times A$.
- Draw a graph with vertices from the set A, and with a directed edge from $u \to v$ if and only if $(u, v) \in R$. What is the maximum indegree of any node? Maximum outdegree?
- Does this relation R have $(x, x) \in R$ for all $x \in A$?
- Is this relation **symmetric**? In other words, does $(x, y) \in R$ imply that $(y, x) \in R$?
- Is this relation **anti-symmetric**? In other words, if $(x, y) \in R$ and $(y, x) \in R$, does it follow that $x = y$?
- Is this relation **transitive**? In other words, if $(x, y) \in R$ and $(y, z) \in R$, does it follow that $(x, z) \in R$?

Relations on sets that are not numbers

Binary relations are subsets of $A \times A$ for some set A.

Let A denote the set of people in the USA.

Consider the following relations on this set:

- Suppose $(a, a') \in R$ if and only if a and a' have the same birthday (not considering the year). What properties does this relation satisfy?
- Suppose $(a, a') \in R$ if and only if a and a' have the same father.
- Suppose $(a, a') \in R$ if and only if a is older than a'.
- Suppose $(a, a') \in R$ if and only if a and a' live in the same city on this date.
- Suppose $(a, a') \in R$ if and only if a doesn't like a'.

What properties do each of these relations satisfy? (Symmetric, anti-symmetric, reflexive, irreflexive, or transitive?)

Partial order

A partial order is a binary relation that is reflexive, anti-symmetric, and transitive.

Standard example:

- R contains pairs (x, y) if and only if x divides y (for positive integers x and y)

Linear order

A linear order is a partial order where all pairs of elements are comparable.

In other words, a linear order a partial order a set A such that $\forall \{x, y\} \subseteq A$, $(x, y) \in R$ or $(y, x) \in R$.

Standard examples

- R contains pairs (x, y) if and only if $x \leq y$ (for real numbers x and y)
- R contains pairs of people (x, y) if and only if x is shorter than y

Equivalence Relations

Remember that an equivalence relation is a binary relation R on set A that is reflexive, symmetric, and transitive.

Think of

- R contains all pairs of people with the same birthday
- R contains all pairs of integers with the same parity
- R contains all pairs of integers with the exact same set of prime factors
- R contains all pairs of graphs with the same number of vertices
- R contains all pairs of functions functions of the integers to the integers that have the same value on input 1.

Equivalence Relations and Equivalence Classes

Remember that an equivalence relation is a binary relation R on set A that is reflexive, symmetric, and transitive.

Given an equivalence relation R, define the **equivalence classes** to be the maximal subsets $A_1, A_2, \ldots,$ of A where all $\{x, y\} \subseteq A_i$ satisfy $(x, y) \in R$.

For example, if A is the set of people and R has pairs (x, y) where x and y have the same birthday, then there are 366 equivalence classes (one for every day).

It is often easier to define an equivalence relation by its equivalence classes!

Class Exercise

Each the following relations is an equivalence relation (prove this at home). In class: determine the number of equivalence classes each has.

- R contains all pairs of positive integers with the same parity
- R contains all pairs of positive integers between 2 and 14 that have the same set of prime factors
- R contains all pairs of finite graphs with the same number of vertices
- R contains all pairs of functions that map the set $\{1, 2, \ldots, 100\}$ to $\{1, 2, 3, 4, 5\}$ that have the same value on input 1.

Equivalence relations and equivalence classes

- R contains all pairs of positive integers with the same parity. *The equivalence classes are $\{1, 3, 5, \ldots\}$ and $\{2, 4, 6, 8, \ldots\}$.*
- R contains all pairs of positive integers between 2 and 14 that have the same set of prime factors. *The equivalence classes are defined by the set of prime factors. So the equivalence classes are $\{2, 4, 8\}, \{3, 9\}, \{6, 12\}, \{5\}, \{7\}, \{10\}, \{11\}, \{13\}$, and $\{14\}$.*
- R contains all pairs of finite graphs with the same number of vertices. *The equivalence classes are defined by the number of vertices. There is an infinite (but countable) number of equivalence classes, since that number can anything in $\{0, 1, 2, \ldots\}$.*
- R contains all pairs of functions that map the set $\{1, 2, \ldots, 100\}$ to $\{1, 2, 3, 4, 5\}$ that have the same value on input 1. *There are five equivalence classes, defined by the value of the function on input 1.*

Functions

A function $f : A \to B$ is actually also just a binary relation, but one with some properties.
To describe f as a binary relation (which we will call R_f), we write it as a relation on $A \cup B$. Then, $(a, b) \in R_f$ if $f(a) = b$.

Note that a function must have some properties:

- $\forall a \in A, \exists b \in B$ such that $(a, b) \in R_f$.
- $\forall a \in A, \forall b \in B, b' \in B$, if $(a, b) \in R_f$ and $(a, b') \in R_f$ then $b = b'$

Note that *this way* of specifying a function – i.e., as a set of ordered pairs – does *not* specify the domain and co-domain. Rosen and other texts require that both be specified, and so this is an incomplete definition.

Drawing binary relations as graphs

When a binary relation R on a set A is finite, it is often easy and helpful to visualize it as a directed graph.

The nodes of the graph are the elements of A, and the elements of R are represented by directed edges. Thus, the element (x, y) is represented by the edge from x to y.

Note that this kind of graph can have self-loops (edges $x \to x$) but not parallel edges (two or more directed edges from x to y).

It is easy to check whether a relation is **reflexive** or **irreflexive**, **symmetric** or **anti-symmetric**, and even **transitive**, when you have its visualization as a graph.

Visualizing functions as directed bipartite graphs

Given a function $F : X \to Y$, if X is finite, we can visualize F as a directed bipartite graph.

The left hand side are the vertices for X, and the right hand side are the vertices in Y.

There is a directed edge from x to $F(x)$ for each $x \in X$.

Note: There must be exactly one edge leaving each $x \in X$ for this graph to represent a function from X to Y.

Questions:

- What will the graph look like when F is $1-1$?
- What will the graph look like when F is onto?

Class Exercise

For each of the functions given below, draw the directed bipartite graph and determine whether the function is 1-1 or onto.

- f is the function from $\{1, 2, \ldots, 5\}$ to $\{3, 4, 5, 6, 7\}$ defined by $f(n) = n + 2$.
- g is the function from $\{1, 2, \ldots, 5\}$ to $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ defined by $g(n) = n + 2$.
- h is the function from $\{1, 2, \ldots, 5\}$ to $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ defined by $h(n) = n \mod 3$.