Preparing for the CS 173 (A) Fall 2018 Midterm 1

1 Basic information

Midterm 1 is scheduled from 7:15-8:30 PM. We recommend you arrive early so that you can start exactly at 7:15. Exams will be collected at 8:30 PM.

There are two rooms for this first midterm: DCL 1320, and ECE 1002. You'll learn which one you are assigned to no later than October 4.

If you are requesting a make-up exam, you should be prepared to do the make-up during the regularly scheduled class period, which is October 11 9:30-10:45 AM. (Right now this is the most likely time for the make-up exam.)

The midterm will be closed book, and will have four parts:

- A proof using weak induction
- A proof using strong induction
- A proof by contradiction
- Several multiple-choice problems

You should bring a pen - not a pencil. We will give as much partial credit as we can, and if you erase something you wrote we can't use it to give you partial credit.

Do not bring your laptop, cellphone, textbook, or notes. This is closed book.

Advice for the midterm. To prepare for the midterm, make sure you can do all problems that appeared in any prior homework or reading quiz, and all the material that was taught in class (even if it did not appear in any homework or reading quiz). In addition, make sure you can do the following set of problems; the midterm is likely to include very similar problems to these.

I have indicated the relative difficulty as follows. If I think the problem is pretty easy, it is not marked. If I think the problem is a bit challenging I mark it with an asterisk (*). If I think the problem might be very challenging, I mark it with two asterisks (**). I strongly recommend you do as many of the easier problems first, before proceeding to the harder problems. Also, do not forget to do problems from each of the sections.

The important thing is learning to write your proofs well, and to understand why they are correct. Here are some guidelines about writing proofs.

1. Be simple where you can be, and in particular please don't use notation where English suffices: all too often the mathematical notation isn't quite right. It's fine to say "let A be the set of integers that are (a) the product of two different prime numbers and (b) are greater than 7" instead of writing it out with set builder notation. If you do want to write it out using formal mathematics, please *also* say in English clearly what you

mean, so you can get credit for what you say in English if what you write in formal mathematics isn't quite right.

2. Justify everything you say, unless it's arithmetic. Justifying what you are saying is one way of avoiding making an incorrect statement! For a proof by induction this means in particular: show where you use the Inductive Hypothesis, and make sure that you use the information you are given in the problem. For example, if you are doing a proof by induction for a recursively defined function, make sure you say that you are using the recursive definition and make sure you are using it properly. On the other hand, you do not need to justify arithmetic.

Try to make sure you understand why your proof is correct. Memorizing a sequence of steps in a proof (e.g., in a weak induction proof to prove a formula) may help you do the same kind of problem but won't help you do something different. (Think, for example, about the challenge in doing the proof by induction that every finite simple graph has an even number of vertices of odd degree. You have to understand why induction works to be able to do that proof, because it doesn't quite have the same structure.) Similarly, proofs by contradiction don't have any specific structure. So try to understand why proofs work rather than memorizing techniques.

Finally, please come to office hours! We will all be very glad to help you directly on any of these problems. You can show your solutions to any of these problems and get feedback from any of us.

2 Practice weak induction problems

1. Let $a(n), n \in \mathbb{Z}^+$ be defined by $a(1) = 0$ and $a(n) = a(n-1)$ if $n \geq 2$. Prove that $a(n) = 0$ for all $n \in \mathbb{Z}^+$.

2. Let $h : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

   - $h(1) = 8$
   - $h(n) = 3h(n-1)$ if $n \geq 2$

   Find a closed form solution for $h(n)$ and prove it correct by induction.

3. Let $F(n)$, $n \in \mathbb{Z}^+$, be a sequence of sets defined by

   - $F(1) = \{1\}$
   - $F(n) = \{n\} \cup F(n-1)$ if $n \geq 2$

   Find a closed form solution for $F(n)$ and prove it correct by induction.

4. Define $f : \mathbb{N} \to \{false, true\}$ such that $f(0) = true$ and $f(n) = \neg f(n-1)$ for all $n > 0$. Prove that $f(n) = true$ if n is even and $f(n) = false$ if n is odd, for all $n \in \mathbb{N}$.

5. (*) Let A_n and B_n, for $n \in \mathbb{Z}^+$, be sets such that $A_n \subseteq B_n$. Prove that $\bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n B_i$ for all $n \in \mathbb{Z}^+$.

6. (*) Let the sets $A_1, A_2, \ldots$ be defined by

   - $A_1 = \{3\}$
   - $A_n = \{x \in \mathbb{Z} | \exists y \in A_{n-1} \text{ such that } x = y + 1\}$, for $n > 1$.

   Figure out what A_2 and A_3 are. Then, find a closed form solution for A_n and prove it correct by induction on n.

7. (**) Let R denote the real numbers, and let function $F_n : R \to R$ be defined recursively for $n = 0, 1, \ldots$, as follows:

   - $F_0(x) = x, \forall x \in R$
   - $F_n(x) = 2F_{n-1}(x), \forall x \in R$, when $n > 0$.

   (a) Show the formulas for $F_1(x), F_2(x)$, and $F_3(x)$, (b) find a closed form formula for $F_n(x)$, and (c) prove your formula correct by induction on n.

3 Practice strong induction problems

1. Let $t(n), n \in \mathbb{Z}^{\geq 0}$, be defined by $t(0) = t(1) = 0$ and $t(n) = t(n-2) + 2$ if $n \geq 2$. Find a closed form solution for $t(n)$ (defined in terms of whether n is even or odd) and prove it by induction.

2. Let $G(n)$, $n \in \mathbb{Z}^+$, be a sequence of sets defined by

   - $G(1) = \{1\}$
   - $G(2) = \{2\}$
   - $G(n) = \{n\} \cup G(n-2)$ if $n \geq 3$

   Find a closed form solution for $G(n)$ and prove it correct by induction.

4 Practice harder induction proofs

Some of these might be able to be solved using weak induction, others might require strong induction, but all of them require coming up with a parameter you can do induction on (and it may not be obvious).

1. (*) Consider the function $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

   - $F(1, x) = F(x, 1) = 1$ for all $x \in \mathbb{Z}^+$ and
   - $F(i, j) = \min\{F(i-1, j), F(i, j-1)\}$ if $i > 1$ and $j > 1$.

   Find a closed form solution for $F(i, j)$ and prove it correct by induction.

2. (*) Consider the function $F : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

- $F(1, m) = F(m, 1) = m$, $\forall m \in \mathbb{Z}^+$
- $F(m, n) = \frac{F(m-1,n)+F(m,n-1)+m+n}{2}$, $\forall (m, n) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $m > 1$ and $n > 1$.

For this function:

- Calculate $F(2, 2), F(3, 2)$, and $F(3, 3)$.
- Come up with a closed form solution for $F(m, n)$ and prove it by induction.

3. (**) Let $A_{i,j}$, where i, j are non-negative integers, be defined by

   - $A_{0,j} = \{2j\}$,
   - $A_{i,0} = \{3i\}$, and
   - $A_{i,j} = A_{i-1,j} \cup A_{i,j-1}$ if i, j are both positive integers.

   For this collection of sets:

   - Determine $A_{1,j}$ for $j = 1, 2, 3, 4, 5$.
   - Find a closed form solution for $A_{1,j}$ and prove it correct by induction.
   - Find a closed form solution for $A_{i,1}$ and prove it correct by induction.
   - Determine $A_{i,j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.
   - Find a closed form solution for $A_{i,j}$, and prove it correct by induction.

4. (**) Suppose $M[x, y]$ is defined for all positive integers x, y by

   - $M[1, x] = x$ for all positive integers x
   - $M[2, x] = 2x$ for all positive integers x
   - $M[k, 1] = k$ for all positive integers k
   - $M[k, x] = \max\{M[k-1, x], M[k, x-1]\}$ for all $x \geq 2$ and integers $k \geq 3$

   Prove or disprove: $M[a, b] \leq \max\{2a, 2b\}$ for all positive integers a, b.

5 Practice proofs by contradiction problems

Some facts you can use use when you do a proof by contradiction:

1. If A is a finite subset of the real numbers, then A has a minimum element and a maximum element.
2. If A is an arbitrary subset of the positive integers, then A has a minimum element.
3. If A is an arbitrary subset of the negative integers, then A has a maximum element.

4. More generally,

   (a) if $N_0 \in \mathbb{R}$ and $A \subseteq \mathbb{Z}^{\geq N_0}$, then A has a minimum element.

   (b) if $N_0 \in \mathbb{R}$ and $A \subseteq \mathbb{Z}^{\leq N_0}$, then A has a maximum element.

Many proofs by contradiction amount to finding the "smallest" counterexample, so you should look at these cases to see if you can use them.

5.1 Do at least two of the induction proofs by contradiction

5.2 Other practice proofs by contradiction

1. Prove by contradiction that there is no largest positive integer.

2. Prove by contradiction that there is no smallest negative integer.

3. Prove by contradiction that there is no smallest positive rational number.

4. Prove by contradiction that there is an infinite number of primes.

5. Prove by contradiction that when p and q are distinct primes, that $\sqrt{pq}$ is not a rational number.

6. (*) We say that a set A is finite if $|A| < \infty$. Prove by contradiction that there is an infinite number of finite subsets of the positive integers.

7. (**) Suppose $P(x)$ is either true or false, depending on the value of x. Now suppose that for all $x \in \mathbb{Z}$, $P(x) \rightarrow P(x+1)$. Prove that the set $\{x \in \mathbb{Z}^+ | P(x)\}$ must be one of the following:

   - $\emptyset$
   - $\mathbb{Z}^+$
   - $S_b = \{x \in \mathbb{Z} | x \geq b\}$, where $b \in \mathbb{Z}^+$

8. (**) Suppose $A \subseteq \mathbb{Z}$ satisfies $x \in A \rightarrow (\exists y \in A, y > x)$. Show that if $A \neq \emptyset$ then A must be infinite.

9. Prove by contradiction that if $x_1, x_2, \ldots, x_n$ is a list of integers whose sum is NOT divisible by y (where y is some integer) then at least one of the x_i is not divisible by y.

10. Prove by contradiction that if n is an integer and $n^3 + 5$ is odd, then n is even.

6 Other problems

The multiple choice problems are likely to be based on problems such as the ones you see below (although these aren't stated as multiple choice problems).

The first set of problems are about logic, sets, and functions. Then there is a separate set of problems about relations and another separate set of problems about graphs.

Problems about logic, sets, and functions.

1. Suppose A is the set of even integers and $B = \{x \in \mathbb{Z} | x^2 \textit{ is even}\}$. Prove or disprove: $A = B$.

2. Prove or disprove: Suppose that A is a subset of the set of positive integers such that $\forall x \in A\ \exists y \in A\ s.t.\ y < x$. Then A must be the empty set.

3. Prove or disprove: Suppose that A is a subset of the set of positive real numbers such that $\forall x \in A\ \exists y \in A\ s.t.\ y < x$. Then A must be the empty set.

4. Prove or disprove: Suppose that A is a subset of the set of positive integers such that $\forall x \in A\ \exists y \in A\ s.t.\ y > x$. Then A must be the empty set.

5. Prove or disprove: Suppose that A is a subset of the set of positive real numbers such that $\forall x \in A\ \exists y \in A\ s.t.\ y > x$. Then A must be the empty set.

6. (*) Consider the following property $P(S)$ about a set S:

   - $\forall x \in \mathbb{Z}^+$, if $x \notin S$ then $\exists y \in S$ such that $y < x$.

   For each of the following sets S, determine if $P(S)$ is true or false.

   (a) $S = \{1, 2, 3\}$
   (b) $S = \{2, 3\}$
   (c) $S = \emptyset$
   (d) $S = \mathbb{Z}^+$
   (e) $S = \{x \in \mathbb{R} | x \geq 3\}$
   (f) $S = \{x \in \mathbb{Z} | x \geq 3\}$
   (g) $S = \{x \in \mathbb{R} | x \leq 3\}$
   (h) $S = \{x \in \mathbb{Z} | x \leq 3\}$
   (i) $S = \{x \in \mathbb{R}^+ | x \leq 3\}$
   (j) $S = \{x \in \mathbb{Z}^+ | x \leq 3\}$
   (k) $S = \{x \in \mathbb{R}^+ | x < 3\}$
   (l) $S = \{x \in \mathbb{Z}^+ | x < 3\}$

(m) $S = \{x \in \mathbb{R}^+ | x > 3\}$

(n) $S = \{x \in \mathbb{Z}^+ | x > 3\}$

7. (*) Suppose $S \subseteq \mathbb{R}$ is a set that satisfies:

   - $\forall \{x, y\} \subset \mathbb{R}, [(x \in S \wedge y > x) \rightarrow y \in S]$

   Which of the following sets could be S? (i.e., which of the following sets satisfy the stated property?):

   - $\emptyset$
   - $\mathbb{Z}$
   - $\mathbb{R}$
   - $\{x \in \mathbb{R} | x > 2\}$
   - $\{x \in \mathbb{R} | x < 2\}$

8. (*) Suppose $S \subseteq \mathbb{R}$ is a set that satisfies:

   - $\forall x \in \mathbb{R}, [x \in S \rightarrow \exists y \in S \ s.t. \ y < x]$

   Which of the following sets could be S? (i.e., which of the following sets satisfy the stated property?):

   - $\emptyset$
   - $\mathbb{Z}$
   - $\mathbb{R}$
   - $\{x \in \mathbb{R} | x > 2\}$
   - $\{x \in \mathbb{R} | x < 2\}$

9. Remember that $\mathcal{P}(S)$ denotes the power set of S (i.e., the set of all subsets of S, see page 121 in Rosen). Consider the following sets:

   - $A = \{S \in \mathcal{P}(\mathbb{Z}) : 1 \notin S\}$,
   - $B = \{S \in \mathcal{P}(\mathbb{Z}) : 1 \in S\}$,
   - $C = \{S \in \mathcal{P}(\mathbb{Z}) : 2 \notin S\}$, and
   - $D = \mathcal{P}(\mathbb{Z})$.

   (a) Draw the Venn Diagram for these four sets, and determine whether each of the areas is empty or non-empty. For each non-empty area, provide an example of an element to prove it's non-empty. For each of the empty areas, prove that it is empty.

   (b) Is $2 \in A$?

   (c) Is $A \cap B = \emptyset$?

10. Suppose we know $P(k) \to P(k+1)$ for all nonnegative integers k. Suppose we also know that $P(7)$ is true and $P(4)$ is false. Use this information to answer the following questions.
    - Do we know what $P(12)$ is? If so, what is it?
    - Do we know what $P(6)$ is? If so, what is it?
    - Do we know what $P(3)$ is? If so, what is it?
11. Negate each of the following logical expressions:
    - $\forall x \in A\ \exists y \in B\ s.t.\ x + y \leq 10$
    - $\exists a \in A\ s.t.\ \forall x \in A, a + x \leq 10$
    - $(P \vee Q) \to (P \wedge \neg Q)$
12. For each of the following, determine if it is a tautology, satisfiable but not a tautology, or always false. Prove your assertion.
    - $(P \vee Q) \to (P \wedge \neg Q)$
    - $P \to \neg P$
    - $(P \to \neg P) \wedge (\neg P \to P)$
    - $(P \to \neg Q) \wedge (\neg Q \to Q)$
    - $(P \to \neg Q) \wedge (\neg Q \to \neg P)$
    - $(P \to \neg Q) \wedge (\neg Q \to P)$
13. (*) Let A be the set of functions $F : \{1, 2, 3\} \to \{1, 2, 3, 4\}$. Let R be the binary relation on A defined by $(f, g) \in R$ if and only if $Image(f) = Image(g)$.
    (a) Give an example of an element of R, and a pair (f, g) of functions such that $(f, g) \notin R$.
    (b) Prove or disprove: R is reflexive, R is irreflexive, R is symmetric, R is anti-symmetric, R is transitive.
14. Define, using set-builder notation, the set of all functions from the reals to the reals that are strictly increasing.
15. Define, using set-builder notation, the set of all functions from the reals to the reals that are 1-1.
16. Define, using set-builder notation, the set of all functions from the integers to the integers that are not onto.
17. Consider the sets
    - $A = \{f : \mathbb{Z}^+ \to \mathbb{Z}^+\}$
    - $B = \{f \in A | Image(f) \neq \mathbb{Z}^+\}$

- $C = \{f \in A | \forall z \in \mathbb{Z}^+, \exists z' \in \mathbb{Z}^+ \ s.t. \ f(z') = z\}$.

Draw the Venn Diagram for these three sets, and prove whether each area is empty or not.

18. Consider the following formula:

$$(A \to B) \land (B \to \neg C) \land (B \to C)$$

- Is this formula a tautology, satisfiable but not a tautology, or not satisfiable? (Prove your answer)
- Rewrite the formula in 2CNF form.

19. (**) Consider the 2CNF formula

$$(\neg A \lor \neg B) \land (B \lor C) \land (\neg C \lor A)$$

- Rewrite it so that it is a conjunction of clauses in the form $X \to Y$.
- Is the formula satisfiable?
- Find a satisfying assignment if the formula is satisfiable.

20. (**) Consider the 2CNF formula given by

$$(\neg A \lor B) \land (\neg B \lor \neg C) \land (\neg B \lor C)$$

- Rewrite it so that it is a conjunction of clauses in the form $X \to Y$.
- Find a satisfying assignment if the formula is satisfiable
- Simplify the formula so that it has no parentheses and no implications ($\to$). Explain your work.

Problems about relations

1. Let A be the set of non-empty subsets of $\mathbb{Z}^+$ and let R be the binary relation on A defined by $(X, Y) \in R$ if and only if $\sum_{x \in X} x = \sum_{y \in Y} y$.
   (a) Give an example of an element of R, and a pair (X, Y) of subsets of $\mathbb{Z}^+$ such that $(X, Y) \notin R$.
   (b) Prove or disprove: R is reflexive, R is irreflexive, R is symmetric, R is anti-symmetric, R is transitive.
   (c) Think about what would change if you had replaced $\sum_{x \in X} x = \sum_{y \in Y} y$ by $\sum_{x \in X} x \leq \sum_{y \in Y} y$.
2. Consider the binary relation R defined on the positive integers by $(x, y) \in R$ if and only if $x + y \leq 100$. Is this relation reflexive? Irreflexive? Symmetric? Anti-symmetric? Transitive? (Prove each of your answers.)

3. Consider the binary relation R defined on the students currently enrolled at the University of Illinois at Urbana-Champaign by $(x, y) \in R$ if and only if x and y were born on the same day of the week (e.g., both born on a Monday, or both born on a Tuesday, etc.). Prove that this relation is an equivalence relation. (Note - this means you should be able to define what an equivalence relation is.) How many equivalence classes does this relation define?

Problems about graphs

1. Consider the graph defined as follows. The vertices correspond to all the people who live in the USA. There is an edge between two distinct vertices if the corresponding students are born on the same day of the week (see previous problem). Is this graph connected? If not, how many components do you think this graph has?

2. Consider the graph formed as follows. There are 8 vertices in the graph, corresponding to the elements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. An edge (i, j) is in the graph if and only if i and j are relatively prime. Draw the graph and then answer the following questions.
   - Is the graph connected?
   - What is the maximum degree of any node?

3. Consider the graph formed as follows. There are 8 vertices in the graph, corresponding to the elements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. An edge (i, j) is in the graph if and only if $i \neq j$ and i and j are *not* relatively prime. Draw the graph and then answer the following questions:
   - Is the graph connected?
   - What is the maximum degree of any node?