CS173
Countability and Cardinality

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CS 173

Cardinality and Countability

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Today

- Cardinality of infinite sets
- Countability and how to prove that a set is countable
- Uncountability, and how to prove that a set is not countable

This material will be on the final exam. (CS 374 assumes you know this material!)

Finite Sets

The **cardinality** of a finite set X is the number of elements in X, and is denoted $|X|$.

Hence, $|\{1, 2, 3, 4, 5\}| = |\{2, 9, 12, 17, 18\}|$.

A set X is **finite** if $|X| = n$ for some $n \in Z$.

Infinite Sets

A set X is **infinite** if there does not exist any $n \in Z$ so that $|X| = n$.

Formal definition: A set X is infinite if $\exists Y \subset X$ (i.e., Y is a proper subset of X) and a 1-1 function $f : X \to Y$.

Examples:

- Let E denote the set of even integers and let $f : \mathbb{Z} \to E$ be defined by $f(x) = 2x$.
- Let $g : \mathbb{Z}^{+} \to \mathbb{Z}^{\geq 5}$ be defined by $g(x) = x + 5$

Each of these is a 1-1 function from a set A to a proper subset of A. Hence the set A is infinite.

We say that $|X| \leq |Y|$ if there is a 1-1 function $g : X \to Y$.

Cardinality of infinite sets

We will say that $|X| = |Y|$ if $\exists f : X \to Y$ where f is a bijection.

A set X is **countably infinite** if there is a bijection from X to $\mathbb{N}$. Using the prior notation, we say X is countably infinite if $|X| = |\mathbb{N}|$.

A set is **countable** if it is finite or countable infinite.

We will prove that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Z}^+|$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$).

Proof that $|\mathbb{Z}| = |\mathbb{N}|$

We prove that $|\mathbb{Z}| = |\mathbb{N}|$ by establishing a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

We will send the non-negative integers to the even natural numbers, and the negative integers to the odd natural numbers.

- $f(x) = 2x$ when $x \geq 0$
- $f(x) = 2|x| - 1$ when $x < 0$

It is clear that f maps integers to natural numbers. To complete the proof:

- We need to prove that f is $1 - 1$
- We need to prove that f is onto

Proving f is $1-1$

Recall $f : \mathbb{Z}$ to $\mathbb{N}$ is defined by

- $f(x) = 2x$ when $x \geq 0$
- $f(x) = 2|x| - 1$ when $x < 0$

We prove that f is $1-1$ by contradiction. If f is not $1-1$, then $\exists \{a, b\} \subset \mathbb{Z}$ such that $f(a) = f(b)$.

Since $f(x)$ is odd if and only if x is negative, it must be that a and b are both negative or both non-negative.

Proving f is $1 - 1$

Case 1: $a, b \geq 0$.

Then $[f(a) = f(b)] \to [2a = 2b] \to [a = b]$.

Case 2: $a, b < 0$.

Then $[f(a) = f(b)] \to [2|a| - 1 = 2|b| - 1] \to [|a| = |b|]$

If both a, b are negative, then $[|a| = |b|] \to [a = b]$.

If $a = 0$ then $|a| = 0$ and so $b = 0$ (and similarly for the case where $b = 0$).

Hence $[f(a) = f(b)] \to [a = b]$ and so f is $1 - 1$.

Proving f is onto

Recall that we need to prove that f is a bijection from $\mathbb{Z}$ to $\mathbb{N}$, where

- $f(x) = 2x$ when $x \geq 0$
- $f(x) = 2|x| - 1$ when $x < 0$

To prove that f is onto we need to show that for any $b \in \mathbb{N}$ there is some $a \in \mathbb{Z}$ such that $f(a) = b$.
Case: b is odd. Then $b = 2x + 1$ for some $x \in \mathbb{Z}^+$.

Let $a = -(x + 1)$. Then

$$f(a) = 2|a| - 1 = 2(x + 1) - 1 = 2x + 1 = b$$

Case: b is even. Then $b = 2x$ for some $x \in \mathbb{Z}^{\geq 0}$. Then

$$f(x) = 2x = b$$

Hence f is onto.

Other bijections

Similarly, you can come up with bijections between every other pair of the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Z}^+$, to prove that they all have the same cardinality.

Note you need to **prove** that the function is a bijection (i.e., that it is 1-1 and onto).

What isn't countable?

Can we prove that some set is **not** countable?

Uncountable sets

A set X is uncountable if X is infinite but $|X| \neq |\mathbb{N}|$.
Examples:

- $[0, 1]$
- $\mathbb{R}$
- $\mathbb{P}(\mathbb{N})$
- The set of functions from $\mathbb{N}$ to $\{0, 1\}$
- The set of all infinite length binary strings

Furthermore, for any set A that is listed above, then

- Any set X that contains A as a subset is uncountable
- Any set X that contains a subset Y where $|Y| = |A|$ is uncountable

Why $\mathbb{P}(\mathbb{N})$ is uncountable

The proof that $\mathbb{P}(\mathbb{N})$ is uncountable is in the book, but we'll go over it here.

Why $\mathbb{P}(\mathbb{N})$ is uncountable

Proof by contradiction.
If $\mathbb{P}(\mathbb{N})$ is countable, then there is a bijection between $\mathbb{P}(\mathbb{N})$ and $\mathbb{N}$, and so we can list these sets $A_0, A_1, A_2, \ldots$.
We will write down these sets in a matrix format with entries 0 and 1, where A_i is represented by i^{th} row.
Hence, $M[i, j] = 1$ if and only if $j \in A_i$.

The matrix M

Recall that $M[i,j] = 1$ if and only if $j \in A_i$.
Example: let's suppose that the first four sets are $A_0 = \{0,3,5\}$, $A_1 = \{2,3\}$, $A_2 = \emptyset$, $A_3 = \{x \in \mathbb{N} : x \geq 3\}$

What do the first four rows of the matrix M look like?

The matrix M

Recall that $M[i, j] = 1$ if and only if $j \in A_i$.

Example: let's suppose that the first four sets are $A_0 = \{0, 3, 5\}$, $A_1 = \{2, 3\}$, $A_2 = \emptyset$, $A_3 = \{x \in \mathbb{N} : x \geq 3\}$

Let's construct $Y \subseteq \{0, 1, 2, 3\}$ so that $i \in Y$ if and only if $i \notin A_i$ for $i = 0, 1, 2, 3$. What is Y?

Diagonalization argument

We prove $\mathbb{P}(\mathbb{N})$ is uncountable using a diagonalization argument.

Consider the infinite matrix representing $\mathbb{P}(\mathbb{N})$.

By construction, every subset of $\mathbb{N}$ is represented by some row in the matrix.

Consider the set Y defined by $j \in Y$ if and only if $M_{j,j} = 0$.

Note that Y is a subset of $\mathbb{N}$.

Finishing the proof

Now we derive the contradiction!

- We assumed that the set $\mathbb{P}(\mathbb{N})$ is countable, and that matrix M has a row for every element in the set.
- We defined the set $Y \in \mathbb{P}(\mathbb{N})$ by $j \in Y$ if and only if $j \notin A_j$ for all $j \in \mathbb{N}$.
- Hence for all $j \in \mathbb{N}$, $Y \neq A_j$.
- Therefore the matrix M cannot have a row for every element of $\mathbb{P}(\mathbb{N})$.
- Hence we derive a contradiction.

Proving a set X is uncountable

To prove a set X is uncountable, do one of the following:

- The same kind of proof by contradiction – enumeration and diagonalization
- Prove that $|X| = |Y|$ where Y is uncountable
- Find an uncountable set Y and show that $Y \subset X$
- Find an uncountable set Y and a 1-1 function from Y to X; this is denoted by $|Y| \leq |X|$

Class Exercise

Prove the set S of infinite length binary strings is uncountable.

Hint: Recall the proof that $\mathbb{P}(\mathbb{N})$ is uncountable.

Suppose S is countable, and then write its matrix representation $M[i, j]$ where the i^{th} row denotes the i^{th} string in S, and $M[i, j]$ is the value (0 or 1) of the j^{th} character in that string.

When is $A \times B$ countable?

Suppose A and B are both countable sets. Is $A \times B$ countable?

Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be the enumeration of these sets.

Can we enumerate this set so that every element appears in some finite index?

When is $A \times B$ countable?

Consider the infinite matrix $M[i, j]$ where $M[i, j]$ corresponds to the ordered pair (a_i, b_j).

Consider the enumeration of the set $A \times B$, given by going down short diagonals (right to left, decreasing):

- $M[1, 1]$
- $M[1, 2], M[2, 1]$
- $M[1, 3], M[2, 2], M[3, 1]$
- $M[1, 4], M[2, 3], M[3, 2], M[4, 1]$
- etc.

Note that every element of $A \times B$ appears at some finite index, and so enumeration defines a bijection between the elements of $A \times B$ and Z^+.

Hence if A and B are countable, then $A \times B$ is countable.

General properties

- If $|X| \leq |Y|$ and Y is countable, then X is countable (recall that $|X| \leq |Y|$ means there is a 1-1 function from X to Y).
- If $X_1, X_2, \ldots, X_k$ are each countable, then $\prod_i X_i$ is countable.
- If $X_1, X_2, \ldots, X_k$ are each countable, then $\cup_i X_i$ is countable.

Hence $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$ are both countable.

Class Exercise

For each of these sets, determine if it is finite, countably infinite, or uncountable.

- $\mathbb{Q}$
- The union of two countable sets
- $\bigcup_{i=1}^{\infty} A_i$ where A_i is finite for all $i \in \mathbb{Z}^+$.
- The set of all finite length binary strings
- The set of functions from A to X, where A is countably infinite and X is finite (e.g., $A = \mathbb{Z}$ and $X = \{1, 2, 3\}$).
- The set of functions from X to A, where A is countably infinite and X is finite (e.g., $A = \mathbb{Z}$ and $X = \{1, 2, 3\}$).
- $\mathbb{P}(Y)$, where Y is a finite set
- $\mathbb{P}(Y)$, where Y is a countably infinite set
- $\mathbb{R}$
- $\mathbb{R} \setminus \mathbb{Q}$

Cantor-Schroeder-Bernstein Theorem

The **Cantor-Schroeder-Bernstein Theorem** theorem shows that for any two sets A, B, $|A| = |B|$ whenever you can find two $1-1$ functions, one from A to B, and the other from B to A.

More specifically, they show that if you have two $1-1$ functions, then there is a *bijection* between the two sets.

Finding two $1-1$ functions is generally easier to do than finding a bijection.

Using Cantor-Schroeder-Bernstein Theorem

For example, to prove $|\mathbb{N}| = |\mathbb{Z}|$, we can write

- $f : \mathbb{N} \to \mathbb{Z}$, where $f(x) = x$
- $g : \mathbb{Z} \to \mathbb{N}$, where
  - $g(x) = 2x$ if $x \geq 0$
  - $g(x) = 2|x| + 1$ if $x < 0$

It's easy to see that f and g are both $1 - 1$, so by the Cantor-Schroeder-Bernstein theorem, $|\mathbb{N}| = |\mathbb{Z}|$.

Cardinality of infinite sets

Consider the binary relation on sets $(X, Y) \in R$ if and only if $|X| = |Y|$.

Note that $|A| = |B|$ and $|B| = |C|$ implies that $|A| = |C|$.

It is easy to see that R is an equivalence relation!

Summary

What we covered today:

- Definition of cardinality for infinite sets
- Definition of countability
- Definition of uncountability
- Diagonalization proofs for uncountability
- Other techniques for proving uncountability
- Cantor-Schroeder-Bernstein Theorem
- How to prove countability